MATH 215 WINTER 2026
Homework Set 0: Geometry Review, §12.1, 12.2

Only some of the questions on this and other homework sets will be graded.
Due January 13, no later than 11:59pm, submitted through Gradescope.

You may work on these problems in groups (in fact, this is encouraged!), *but you must submit your own set of solutions*. The purpose of the homework is not (only) to get the right answer, but also to clearly and concisely explain your work in a way that another person can understand it. Illegible, unclear, or otherwise unintelligible answers will receive no credit.

Question 1: In this problem you will explain using simple plane geometry how to compute the area of a parallelogram in terms of Cartesian coordinates.[1] Let P be a parallelogram with vertices at $(0,0)$, (a,b), $(a+c,b+d)$, and (c,d) (see Figure 1). For this problem, you may take as a given that the area of a parallelogram is equal to its base multiplied by its height.

(a) Assuming $a,b,c,d>0$ and $ad>bc$, prove that the area of P is $ad-bc$ by cutting P into pieces and reassembling them into a parallelogram whose area is easier to compute (see Figure 1; in the middle figure the dashed horizontal lines are parallel to the x-axis). Where in your argument did you use that $ad>bc$?

(b) Suppose $a,b,c,d>0$ and $ad<bc$. Why is it that the answer from part (a) *cannot* be true? What should the area be? Why? How is this case geometrically different from part (a)?

(c) What geometric information do you have about the figure P in the case $ad=bc$?

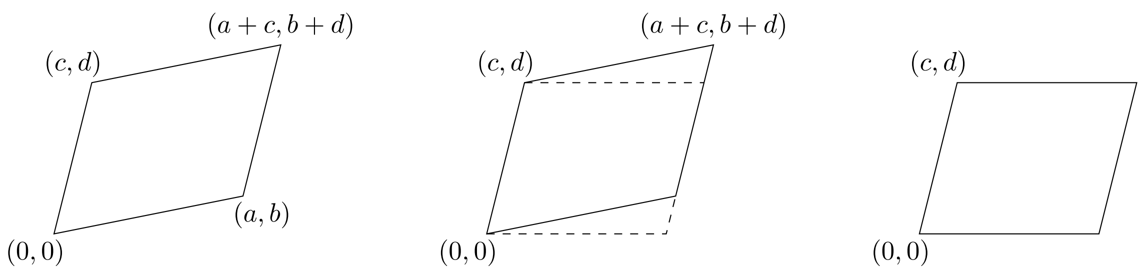


Figure 1: Idea for a proof for Problem 1: The first and third parallelograms have the same area – make sure you justify why. The lower left vertex is at $(0,0)$ in all the figures. What are the coordinates of all the vertices in the second and third pictures? What is the base and height of the third parallelogram?

Question 2: Find the equations of the following spheres in $\mathbb{R}^3$:

(a) centered at $(-1,3,2)$ with radius 4

(b) with center on the negative y-axis and tangent to the planes $x=3$ and $y=-4$

(c) having the circle $x^2+y^2-2x-4y-4=0$ in the plane $z=2$ as a great circle[2]. *Note:* A *great circle* is the intersection of a sphere with a plane containing the center of the sphere. Put another way, it is a circle with the same radius and center as the sphere.

Question 3: As minimally as you can, navigate from the origin to the point $(-1,2)$ using only the vectors:

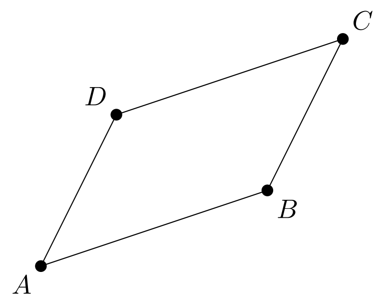
(a) $\mathbf{v}_1=\langle 1,\ 1\rangle$ and $\mathbf{v}_2=\langle 1,\ -1\rangle$

(b) $\mathbf{u}_1=\langle 2,\ 1\rangle$ and $\mathbf{u}_2=\langle 1,\ 1\rangle$

(c) $\mathbf{w}_1=\langle 2,\ 1\rangle$ and $\mathbf{w}_2=\langle 1,\ -1\rangle$

[1] You may not use cross products to solve this problem.
[2] This is sometimes called an *orthodrome*.

Question 4: Sketch a parallelogram and label its vertices as in the figure below, and carefully construct arrows representing the following vectors. Show your results for each item in a small sketch. (Graph paper may be helpful.)

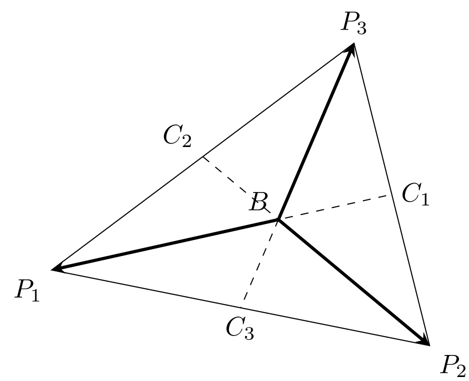


(a) $\overrightarrow{DB} - \overrightarrow{DC}$

(b) $\overrightarrow{AD} + \overrightarrow{BC}$

(c) $\overrightarrow{AC} - \overrightarrow{DB}$

Question 5: Consider a triangle with vertices P_1, P_2, P_3 (see figure below).



Define a point B by requiring that

$$\overrightarrow{P_3B} = \frac{1}{3}\left(\overrightarrow{P_3P_1} + \overrightarrow{P_3P_2}\right)$$

(a) Let C_3 be the midpoint of side P_1P_2. Show that

$$\overrightarrow{P_3B} = \frac{2}{3}\overrightarrow{P_3C_3}$$

(b) Show that, for the *same* point B as above, one has

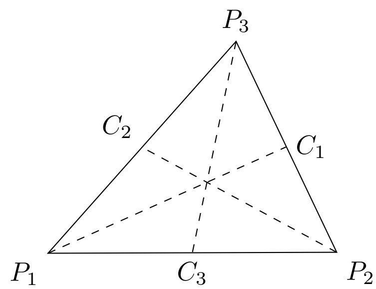
$$\overrightarrow{P_2B} = \frac{1}{3}\left(\overrightarrow{P_2P_1} + \overrightarrow{P_2P_3}\right)$$

and that therefore B is the intersection of the three medians of the triangle.

(c) Show that $\overrightarrow{BP_1} + \overrightarrow{BP_2} + \overrightarrow{BP_3} = \vec{0}$. (This last property says that B is the *centroid* of the triangle. The notion of a centroid, and some variations on it, will appear later in the term, and possibly even in real life.)

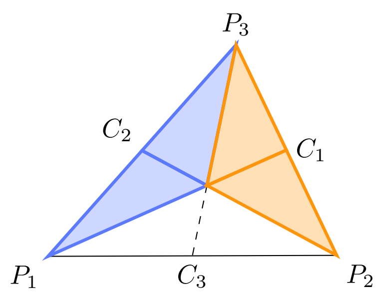
Question 6: Let's revisit the last question, and prove a cool fact about this triangle.

Consider a triangle with vertices P_1, P_2, P_3. Without loss of generality, suppose P_1 is at the origin and P_2 lies on the x-axis.



Constructing the three medians of this triangle splits our triangle into six smaller triangles. In the last problem you showed that the centroid is the intersection of the medians, but you also showed that the centroid splits each median into a long part and a short part, and that the long part is exactly twice as long as the short part (go back and check to see that this is true).

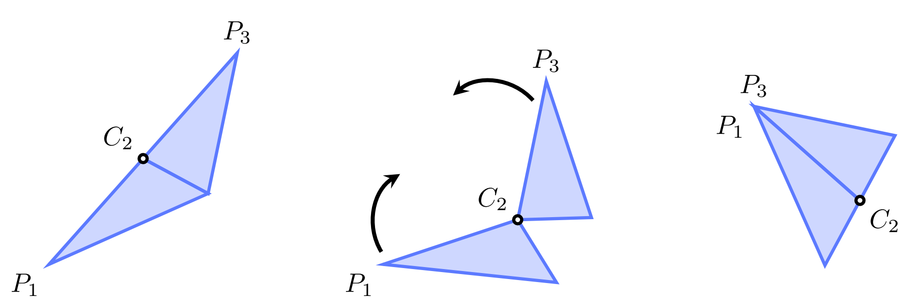
Let's look at the six smaller triangles formed by the medians. These six triangle form three pairs, with each pair sharing a vertex at one of the midpoints. In the picture below, we shade the triangles according to which pair we should group them into:



We see there is a blue pair, an orange pair, and a white pair. The blue pair share a vertex at C_2, the orange pair share a vertex at C_1, and the white pair share a vertex at C_3. Here is what we are going to do, using the blue pair as an example.

- Pretend there is a hinge at C_2.
- Rotate P_1 clockwise and P_3 counterclockwise until they meet. Because C_2 is the midpoint between P_1 and P_3, this will form a new triangle.

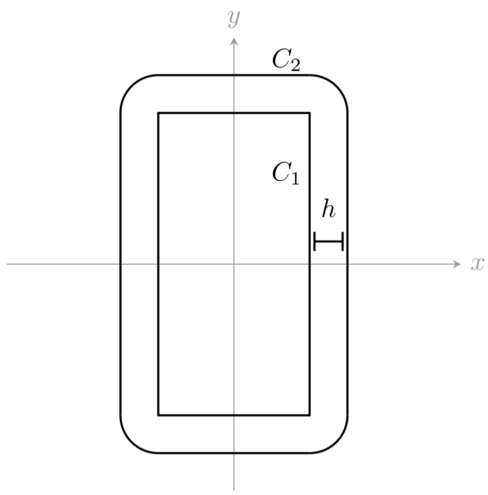
I have attempted to indicate this in a diagram:



You can do this procedure for each of the three pairs of triangle. Your job is to show that the three resultant triangles are *the same*. (Not just congruent, but actually the same.) There are many different ways to prove this, and I want to leave the method open to you. If you want to choose coordinates for the original points and work from that, you can, and you can leverage the power that a coordinate system and vectors gives us to find the relevant lengths and angles. There are also at least two purely geometric solutions that I know of, and I encourage you to try to find one of them. The amazing this about this problem is that if you did it a second time (i.e. on this new triangle) you would get back a scaled down version of our original triangle!

Extra Credit: This question explores some ideas that are going to come up waaaaaaaay at the end of the course, but we can still use a bit of basic geometry to get the right geometric idea in a few special cases.

(a) Let C_1 be the rectangle with vertices $(1, 2)$, $(-1, 2)$, $(-1, -2)$, and $(1, -2)$, and let C_2 be the "rectangle" with rounded corners that is always a distance h away from C_1 (see figure below). Find the area between C_1 and C_2. *Note:* The rounded corners of C_2 are circular arcs that are a distance h away from the nearest corner of C_1.



(b) Write the length of C_2 (the "perimeter" of the rounded rectangle) as the length of C_1 plus a corrective term. Explain how you find the corrective term.

(c) Repeat the previous two parts for a new pair of curves, where C_1 is the circle of radius R centered at the origin, and C_2 is the circle of radius $R + h$ centered at the origin.

(d) Do you think your answer to the previous part generalizes if we allow C_1 to be an arbitrary, but still "nice", planar curve, and we keep the restriction that C_2 can be formed by tracing out a curve around C_1 which is a constant width h away? What might you expect to find?

(e) Let's now turn to slightly different, but related question. For this problem we should note that a curve is *closed* if it begins where it ends, and a curve is *simple* if it does not intersect itself. Suppose you are riding a bike of body-length b, and you follow a path such that the front wheel of the bike traces out a simple, closed curve. (Nothing too wiggly – the rear wheel of the bike never intersects the path of the front wheel of the bike.) Necessarily, the rear wheel of the bike also traces out a related, but distinct, simple closed curve. Show that the area between these two curves is a constant that does not depend on the length of the curves. How does this relate to your answers to the previous parts? Intuitive (i.e. non-technical) explanations are welcome.