MATH 215 WINTER 2026
Homework Set 0: Minimal Solutions

A possible solution: In what follows, we present a set of minimal solutions to the homework. These are not the only solutions, and we don't always provide enough detail to fully justify the work–we provide enough detail for you to understand the solution and fill in some of the details yourself.

Question 1: In this problem you will explain using simple plane geometry how to compute the area of a parallelogram in terms of Cartesian coordinates.[1] Let P be a parallelogram with vertices at $(0, 0)$, (a, b), $(a + c, b + d)$, and (c, d) (see Figure 1). For those problem, you may take as a given that the area of a parallelogram is equal to its base multiplied by its height.

(a) Assuming $a, b, c, d > 0$ and $ad > bc$, prove that the area of P is $ad - bc$ by cutting P into pieces and reassembling them into a parallelogram whose area is easier to compute (see Figure 1; in the middle figure the dashed horizontal lines are parallel to the x-axis). Where in your argument did you use that $ad > bc$?

A possible solution: First notice that we should establish that the "top" triangle in the middle panel of Figure 1 is the same as the "bottom" triangle. However, this is simple enough to do using basic geometry. Because we begin with a parallelogram, the two dashed horizontal lines are the same length–the horizontal shift in the left endpoints of the two lines is balanced by the same shift in the right points of the lines. The same reasoning argues that the short sides of both triangles is the same. The longest side of the triangles are parallel sides of the parallelogram, and so have the same length. Similarly, each of angles of the triangles can be shown to be equivalent by an appropriate application of cutting two parallel lines by a transverse line. Thus, the area of the top triangle is the same as the area of the bottom triangle, and the area of the parallelogram in the rightmost panel is the same as the area of the parallelogram in the leftmost panel.

Now we focus on the rightmost parallelogram in Figure 1. The height is d, self-evidently. To find the base, we notice that the slope of the line of the right edge is d/c, and the right edge contains the point (a, b). All we need do is find the coordinate of the lower right corner of the parallelogram, which has coordinate $(x, 0)$. You can now use any method of finding a point on a line given that we know one point and the slope. For example, we must have

$$\frac{d}{c} = \frac{b - 0}{a - x} \qquad \Rightarrow \qquad x = a - \frac{bc}{d}$$

The area of our parallelogram is then

$$\text{base} \cdot \text{height} = x \cdot d = ad - bc,$$

as expected.

Where did we use the fact that $ad > bc$? This is built into the picture we used in Figure 1. The condition $ad > bc$ is equivalent to the slope of the bottom edge of the parallelogram being less than the slope of the left edge of the parallelogram. If this condition were not met, the picture we began with would be wrong, and there would need to be some changes in the calculation (which we will see in the following parts).

[1] You may not use cross products to solve this problem.

(b) Suppose $a, b, c, d > 0$ and $ad < bc$. Why is it that the answer from part (a) *cannot* be true? What should the area be? Why? How is this case geometrically different from part (a)?

A possible solution: At first glance, we see the answer from (a) cannot possibly be correct if $ad < bc$, because our answer would give a negative number for the area. However, the mere fact that our answer is nonsensical is not enough to dismiss it out of hand–a number being negative is not *a priori* a bad thing, but in this case it does indicate that a problem has arisen somewhere. That is to say, it means we have missed something! Our goal is discover what it is that we have missed.

In this particular case, notice that $ad > bc$ is equivalent to saying the slope of the line connecting $(0, 0)$ to (a, b) is less than the slope of the line connecting $(0, 0)$ to (c, d). If $ad < bc$, the relationship between these slopes is reversed, and the picture we drew is no longer correct–this is what was missed. The picture we gave for the hint has a hidden assumption about the relative sizes of some of the numbers in this problem, and that assumption implies $ad > bc$. In order to allow this new condition, we would need to redraw our picture consistent with our assumptions. If we did this and repeated the relevant calculation from the last part, we will find the area is $bc - ad$, what we expect.

In general, one can show that any parallelogram with vertices at $(0, 0)$, (a, b), (c, d), and $(a + c, b + d)$, irrespective of the signs of a, b, c, and d, will have an area of $|ad - bc|$.

(c) What geometric information do you have about the figure P in the case $ad = bc$?

A possible solution: In this case, our earlier work would show that the area is zero. Does this make sense? If $ad = bc$, then we also have $d/c = b/a$, which indicates the slopes of the lines forming the parallelogram are the same, meaning that we don't actually have a parallelogram–we have a line! As the area of a line is zero, our answer makes sense.

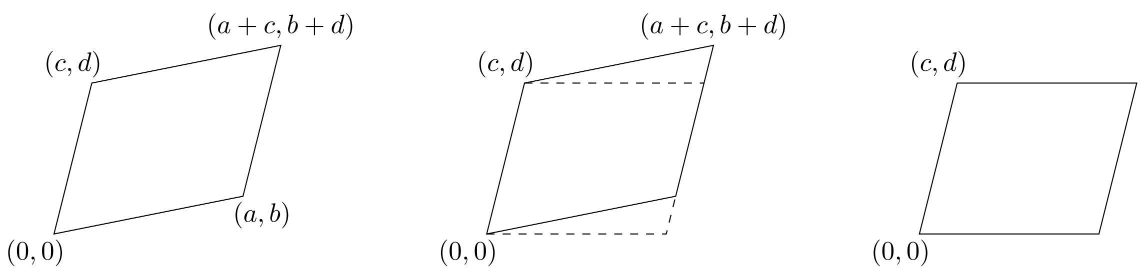


Figure 1: Idea for a proof for Problem 1: The first and third parallelograms have the same area – make sure you justify why. The lower left vertex is at $(0, 0)$ in all the figures. What are the coordinates of all the vertices in the second and third pictures? What is the base and height of the third parallelogram?

Question 2: Find the equations of the following spheres in $\mathbb{R}^3$:

(a) centered at $(-1, 3, 2)$ with radius 4

> *A possible solution:* The equation of a sphere with center at (x_0, y_0, z_0) and radius r is given by
>
> $$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$
>
> For this part, we must have
>
> $$(x - (-1))^2 + (y - 3)^2 + (z - 2)^2 = 4^2$$
>
> or, slightly simplified,
>
> $$(x + 1)^2 + (y - 3)^2 + (z - 2)^2 = 16$$

(b) with center on the negative y-axis and tangent to the planes $x = 3$ and $y = -4$

> *A possible solution:* Because the center is on the negative y-axis, we know the coordinate of the center of the sphere is of the form $(0, y_0, 0)$, where $y_0 < 0$.
>
> Next, we see that the plane $x = 3$ is a distance of 3 from the yz-plane, and therefore from the y-axis. Because our sphere is tangent to this plane, it must be that the radius of the sphere is 3.
>
> All that is left is to find the center of the sphere. Because the sphere is tangent to the plane $y = -4$ and has radius 3, we need the coordinate y_0 to be a distance 3 from $y = -4$. That means we can either have $y_0 = -1$ or $y_0 = -7$. Thus, our sphere could either be
>
> $$x^2 + (y + 1)^2 + z^2 = 9$$
>
> or
>
> $$x^2 + (y + 7)^2 + z^2 = 9$$

(c) having the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ in the plane $z = 2$ as a great circle[2]. *Note:* A *great circle* is the intersection of a sphere with a plane containing the center of the sphere. Put another way, it is a circle with the same radius and center as the sphere.

> *A possible solution:* Because the given circle is a great circle, both the radius and the center of the sphere coincide with the radius and the center of the circle. The given circle can be simplified by completing the square for both $x^2 - 2x$ and $y^2 - 4y$, giving
>
> $$x^2 + y^2 - 2x - 4y - 4 = (x - 1)^2 + (y - 2)^2 - 9 = 0$$
>
> Thus, the equation of the circle is clearly written as
>
> $$(x - 1)^2 + (y - 2)^2 = 9,$$
>
> giving a radius of 3. Because this circle lies in the plane $z = 2$, the center of the circle is at $(1, 2, 2)$. Thus, the equation of the sphere must be
>
> $$(x - 1)^2 + (y - 2)^2 + (z - 2)^2 = 9$$

[2]This is sometimes called an *orthodrome*.

Question 3: As minimally as you can, navigate from the origin to the point $(-1, 2)$ using only the vectors:

(a) $\mathbf{v}_1 = \langle 1,\, 1\rangle$ and $\mathbf{v}_2 = \langle 1,\, -1\rangle$

A possible solution: You can approach this problem either geometrically or algebraically, but it is good to be able to do either.

Algebraically, we want to find real numbers a and b such that

$$a\mathbf{v}_1 + b\mathbf{v}_2 = \langle -1,\, 2\rangle$$

Put another way, we want to solve the system of equations

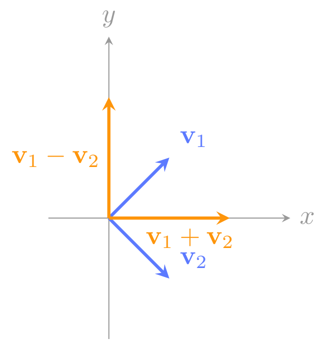
$$a\,\langle 1,\, 1\rangle + b\,\langle 1,\, -1\rangle = \langle -1,\, 2\rangle$$

Writing out explicitly the equation for each component, we have

$$a + b = -1, \qquad a - b = 2$$

Some light algebra will give you $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

Geometrically, we notice that $\mathbf{v}_1 + \mathbf{v}_2 = \langle 2,\, 0\rangle$ and $\mathbf{v}_1 - \mathbf{v}_2 = \langle 0,\, 2\rangle$.



This means that taking a step in the positive x-direction is equivalent to $\frac{1}{2}\,(\mathbf{v}_1 + \mathbf{v}_2)$ and taking a step in the right direction is equivalent to $\frac{1}{2}\,(\mathbf{v}_1 - \mathbf{v}_2)$.

Because we want to take 1 step to the left and 2 steps up, we have

$$\langle -1,\, 2\rangle = (-1)\cdot\frac{1}{2}\,(\mathbf{v}_1 + \mathbf{v}_2) + 2\cdot\frac{1}{2}\,(\mathbf{v}_1 - \mathbf{v}_2) = \frac{1}{2}\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2$$

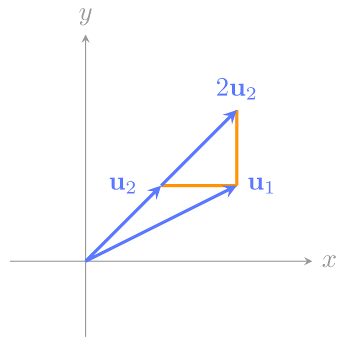
as expected.

(b) $\mathbf{u}_1 = \langle 2,\, 1\rangle$ and $\mathbf{u}_2 = \langle 1,\, 1\rangle$

A possible solution: Again, you can solve this system algebraically to quickly find

$$a\mathbf{u}_1 + b\mathbf{u}_2 = \langle -1,\, 2\rangle \qquad \Rightarrow \qquad 2a + b = -1, \quad a + b = 2 \qquad \Rightarrow \qquad a = -3,\, b = 5$$

More interestingly, geometrically we can see that $\mathbf{u}_1 - \mathbf{u}_2 = \langle 1,\, 0\rangle$ and $2\mathbf{u}_2 - \mathbf{u}_1 = \langle 0,\, 1\rangle$. Below we give a picture of $\mathbf{u}_1$, $\mathbf{u}_2$, and $2\mathbf{u}_2$, and show the right triangle hidden in there:



We now know every step to the right is equivalent to $\mathbf{u}_1 - \mathbf{u}_2$, and every step up is equivalent to $2\mathbf{u}_2 - \mathbf{u}_1$. One step to the left and two steps up is then given by

$$\langle -1,\, 2\rangle = -1\,(\mathbf{u}_1 - \mathbf{u}_2) + 2\,(2\mathbf{u}_2 - \mathbf{u}_1) = -3\mathbf{u}_1 + 5\mathbf{u}_2$$

(c) $\mathbf{w}_1 = \langle 2,\, 1\rangle$ and $\mathbf{w}_2 = \langle 1,\, -1\rangle$

A possible solution: The exact same methods we've used in the previous two parts will follow. We see that

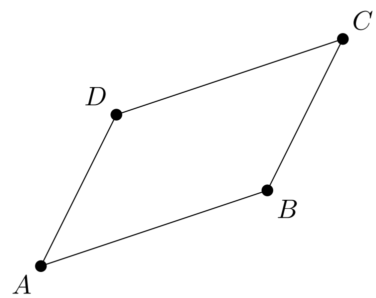
$$\mathbf{w}_1 + \mathbf{w}_2 = \langle 3,\, 0\rangle \qquad \text{and} \qquad \mathbf{w}_1 - 2\mathbf{w}_2 = \langle 0,\, 3\rangle$$

Then one step to the left and two steps up gives

$$\langle -1,\, 2\rangle = -1 \cdot \frac{1}{3}\,(\mathbf{w}_1 + \mathbf{w}_2) + 2 \cdot \frac{1}{3}\,(\mathbf{w}_1 - 2\mathbf{w}_2) = \frac{1}{3}\mathbf{w}_1 - \frac{5}{3}\mathbf{w}_2$$

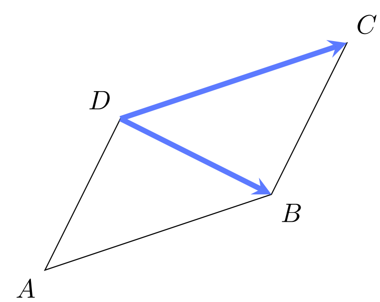
In all three of these problems, notice that our solution method was to reconstruct the standard basis vectors from the given vectors. That is, as long as the vectors we began with were not parallel, we were always able to construct a pair of orthogonal vectors and use those to navigate space. This is a big idea in math (and especially in linear algebra), and one that we will explore as the class progresses.

Question 4: Sketch a parallelogram and label its vertices as in the figure below, and carefully construct arrows representing the following vectors. Show your results for each item in a small sketch. (Graph paper may be helpful.)

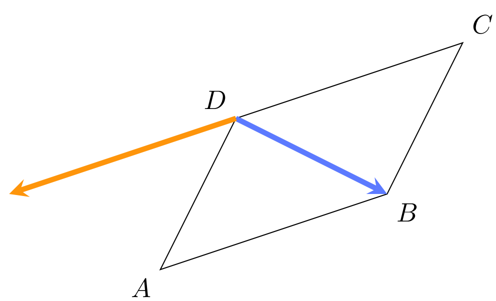


(a) $\overrightarrow{DB} - \overrightarrow{DC}$

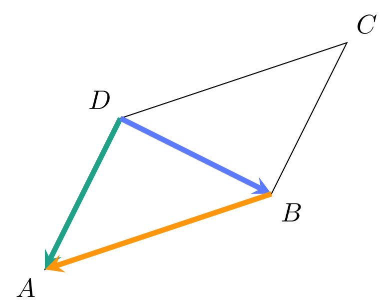
A possible solution: First, draw the two given vectors:



Next, change the direction on $\overrightarrow{DC}$ because we are subtracting it:



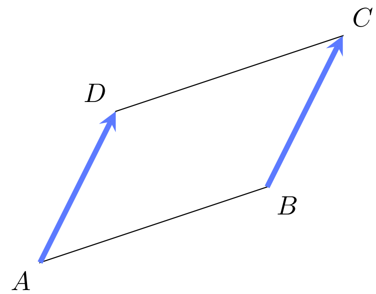
Now, move $-\overrightarrow{DC}$ to the end of $\overrightarrow{DB}$ to add them:



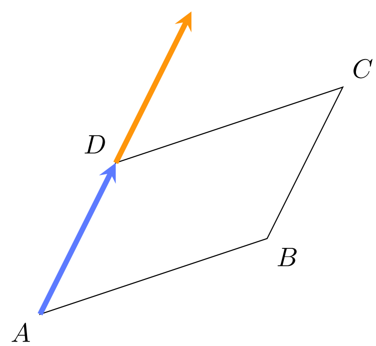
Notice that the resulting vector is simply $\overrightarrow{DA}$.

(b) $\overrightarrow{AD} + \overrightarrow{BC}$

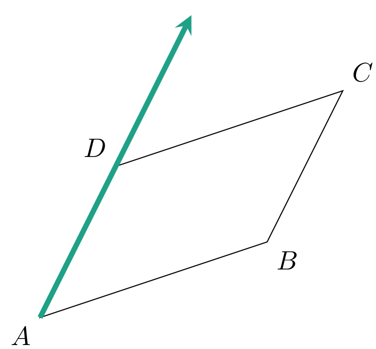
A possible solution: First, draw the two given vectors:



Shift $\overrightarrow{BC}$ to the tip of $\overrightarrow{AD}$ to add them:

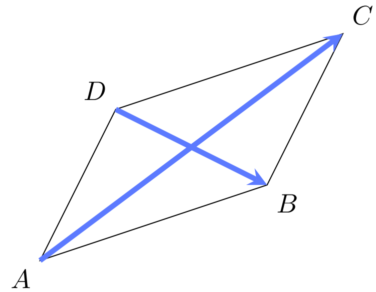


See that this is equivalent to the vector $2\overrightarrow{AD}$:

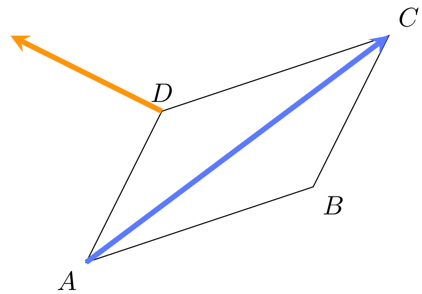


(c) $\overrightarrow{AC} - \overrightarrow{DB}$

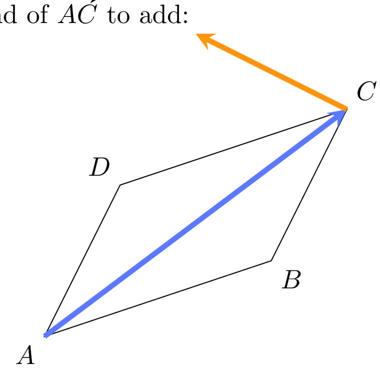
A possible solution: Draw the two given vectors:



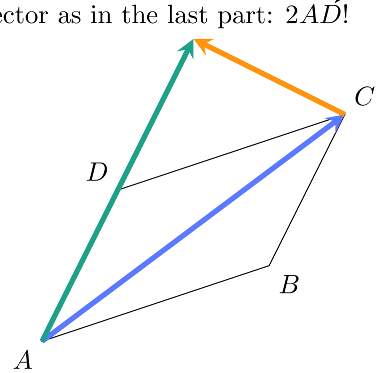
Switch the direction of $\overrightarrow{DB}$ since we are subtracting it:



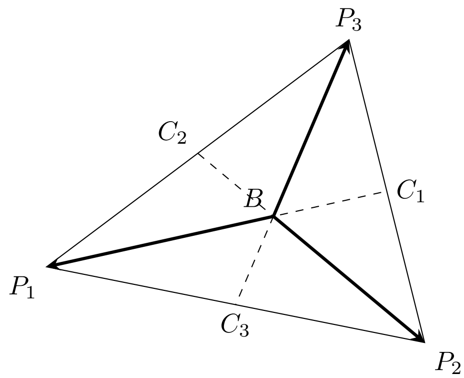
Move the vector $\overrightarrow{BD}$ to the end of $\overrightarrow{AC}$ to add:



Notice that this is the same vector as in the last part: $2\overrightarrow{AD}$!



Question 5: Consider a triangle with vertices P_1, P_2, P_3 (see figure below).



Define a point B by requiring that

$$\overrightarrow{P_3B} = \frac{1}{3}\left(\overrightarrow{P_3P_1} + \overrightarrow{P_3P_2}\right)$$

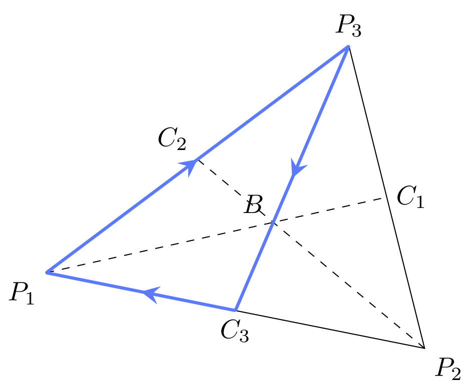
(a) Let C_3 be the midpoint of side P_1P_2. Show that

$$\overrightarrow{P_3B} = \frac{2}{3}\overrightarrow{P_3C_3}$$

A possible solution: First notice that

$$\overrightarrow{P_3C_3} + \overrightarrow{C_3P_1} + \overrightarrow{P_1P_3} = \mathbf{0}$$

This is true because following this path ends at the same point it begins (marked in blue in the diagram below):



An entirely analogous argument will show that

$$\overrightarrow{P_3C_3} + \overrightarrow{C_3P_2} + \overrightarrow{P_2P_3} = \mathbf{0}$$

Next, notice that $\overrightarrow{C_3P_2} = -\overrightarrow{C_3P_1}$. Because C_3 is the midpoint between P_1 and P_2, the magnitudes of these two vectors are the same, and because they lie on the same line emanating from the same point in opposite directions, we have the desired equality.

Add together the two equations that are both equal to zero, and cancelling out appropriately, we have

$$2\overrightarrow{P_3C_3} + \overrightarrow{P_1P_3} + \overrightarrow{P_2P_3} = \mathbf{0} \qquad \Rightarrow \qquad 2\overrightarrow{P_3C_3} = \overrightarrow{P_3P_1} + \overrightarrow{P_3P_2}$$

Putting this relationship into the defining equation for B, we have

$$\overrightarrow{P_3B} = \frac{1}{3}\left(\overrightarrow{P_3P_1} + \overrightarrow{P_3P_2}\right) = \frac{1}{3}\left(2\overrightarrow{P_3C_3}\right),$$

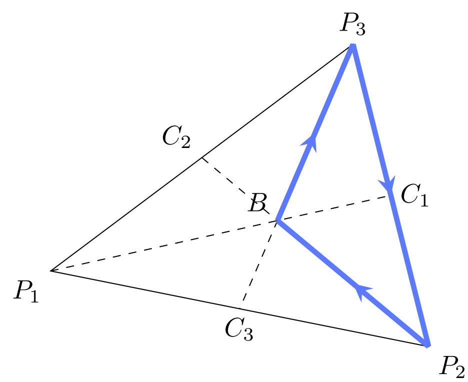
which is the desired relationship.

(b) Show that, for the *same* point B as above, one has

$$\overrightarrow{P_2B} = \frac{1}{3}\left(\overrightarrow{P_2P_1} + \overrightarrow{P_2P_3}\right)$$

and that therefore B is the intersection of the three medians of the triangle.

A possible solution: We are going to apply the same basic trick as last time, and traverse the given triangle along a few different paths. First, we follow the path $P_2 \to B \to P_3 \to P_2$, marked below:



Because this path ends where it begins, the associated vector addition will be zero:

$$\overrightarrow{P_2B} + \overrightarrow{BP_3} + \overrightarrow{P_3P_2} = \mathbf{0} \qquad \Rightarrow \qquad \overrightarrow{P_2B} = \overrightarrow{P_3B} + \overrightarrow{P_2P_3}$$

Applying the result from the previous part we find

$$\overrightarrow{P_2B} = \frac{1}{3}\left(\overrightarrow{P_3P_1} + \overrightarrow{P_3P_2}\right) + \overrightarrow{P_2P_3} = \frac{2}{3}\overrightarrow{P_2P_3} + \frac{1}{3}\overrightarrow{P_3P_1}$$

Next, notice that a complete trip around the triangle will imply

$$\overrightarrow{P_3P_1} + \overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} = \mathbf{0} \qquad \Rightarrow \qquad \overrightarrow{P_3P_1} = \overrightarrow{P_3P_2} + \overrightarrow{P_2P_1}$$

Putting this result into the previous equation we find

$$\overrightarrow{P_2B} = \frac{2}{3}\overrightarrow{P_2P_3} + \frac{1}{3}\left(\overrightarrow{P_3P_2} + \overrightarrow{P_2P_1}\right) = \frac{1}{3}\left(\overrightarrow{P_2P_3} + \overrightarrow{P_2P_1}\right),$$

which is the desired result for $\overrightarrow{P_2B}$.

At this point, we can now apply the result from part (a) to arrive at the conclusion that $\overrightarrow{P_2B} = \frac{2}{3}\overrightarrow{P_2C_2}$.

Similarly, we can reapply the above argument, more or less identically, to the vector $\overrightarrow{P_1B}$ and arrive at a similar conclusion that $\overrightarrow{P_1B} = \frac{2}{3}\overrightarrow{P_1C_1}$.

We have now established that the point B lies on each of the lines $\overrightarrow{P_jC_j}$, and thus is the intersection of the three medians.

(c) Show that $\overrightarrow{BP_1} + \overrightarrow{BP_2} + \overrightarrow{BP_3} = \vec{0}$. (This last property says that B is the *centroid* of the triangle. The notion of a centroid, and some variations on it, will appear later in the term, and possibly even in real life.)

A possible solution: The results from parts (a) and (b) give us

$$3\overrightarrow{BP_1} = \overrightarrow{P_1P_2} + \overrightarrow{P_1P_3}$$
$$3\overrightarrow{BP_2} = \overrightarrow{P_2P_1} + \overrightarrow{P_2P_3}$$
$$3\overrightarrow{BP_3} = \overrightarrow{P_3P_1} + \overrightarrow{P_3P_2}$$

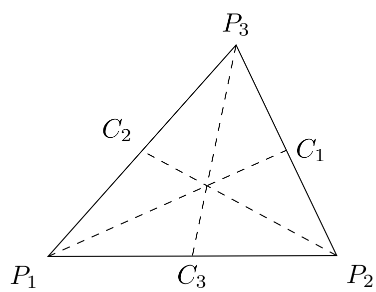
Adding each of these up, we will find

$$3\left(\overrightarrow{BP_1} + \overrightarrow{BP_2} + \overrightarrow{BP_3}\right) = \left(\overrightarrow{P_1P_2} + \overrightarrow{P_2P_1}\right) + \left(\overrightarrow{P_1P_3} + \overrightarrow{P_3P_1}\right) + \left(\overrightarrow{P_2P_3} + \overrightarrow{P_3P_2}\right) = \mathbf{0}$$

where we have used the fact that each pair $\overrightarrow{P_jP_k} + \overrightarrow{P_kP_j} = \mathbf{0}$ independently.

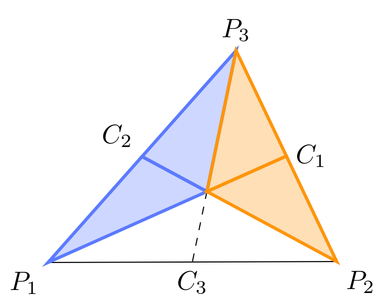
Question 6: Let's revisit the last question, and prove a cool fact about this triangle.

Consider a triangle with vertices P_1, P_2, P_3. Without loss of generality, suppose P_1 is at the origin and P_2 lies on the x-axis.



Constructing the three medians of this triangle splits our triangle into six smaller triangles. In the last problem you showed that the centroid is the intersection of the medians, but you also showed that the centroid splits each median into a long part and a short part, and that the long part is exactly twice as long as the short part (go back and check to see that this is true).

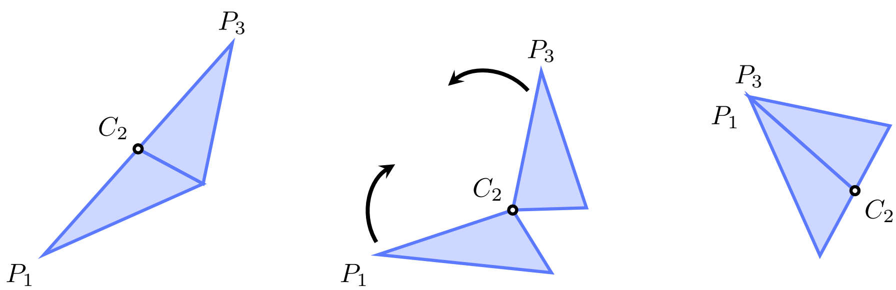
Let's look at the six smaller triangles formed by the medians. These six triangle form three pairs, with each pair sharing a vertex at one of the midpoints. In the picture below, we shade the triangles according to which pair we should group them into:



We see there is a blue pair, an orange pair, and a white pair. The blue pair share a vertex at C_2, the orange pair share a vertex at C_1, and the white pair share a vertex at C_3. Here is what we are going to do, using the blue pair as an example.

- Pretend there is a hinge at C_2.
- Rotate P_1 clockwise and P_3 counterclockwise until they meet. Because C_2 is the midpoint between P_1 and P_3, this will form a new triangle.

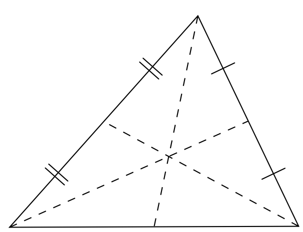
I have attempted to indicate this in a diagram:



You can do this procedure for each of the three pairs of triangle. Your job is to show that the three resultant triangles are *the same*. (Not just congruent, but actually the same.) There are many different ways to prove this, and I want to leave the method open to you. If you want to choose coordinates for the original points and work from that, you can, and you can leverage the power that a coordinate system and vectors gives us to find the relevant lengths and angles. There are also at least two purely geometric solutions that I know of, and I encourage you to try to find one of them. The amazing this about this problem is that if you did it a second time (i.e. on this new triangle) you would get back a scaled down version of our original triangle!

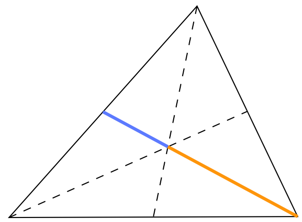
A possible solution: There are so many cool ways to solve this problem I almost don't want to show you, so you can play around with this some more. Let's give a sketch of the idea. We will show that the blue triangle is the same as the orange triangle, and leave it to you to justify that the same is true for the third triangle.

First, let's note some convenient facts. Because the C's are midpoints, we have that each side of the triangle is split into two equal parts:

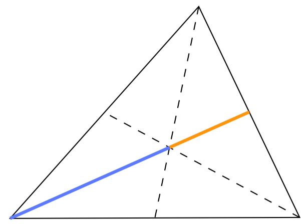


This helps establish that our rotation procedure will in general generate a triangle (i.e. the endpoints will actually meet).

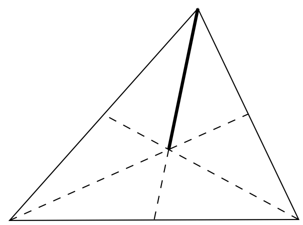
Next, recall that in the last problem you proved that the short side of a median is half as long as the long side of any median, where the "short" side and "long" side are determined by the centroid. That is, in the picture below the blue line segment is half as long as the orange line segment. When the triangle is folded, the blue segment gets doubled to become one side of the blue triangle, while the orange line segment also forms a single side of the orange triangle.



Similarly, in the picture below, the orange line segment is half as long as the blue line segment, and the same argument applies as above.



Finally, the two triangles share a side formed by the thick black line in the picture below:



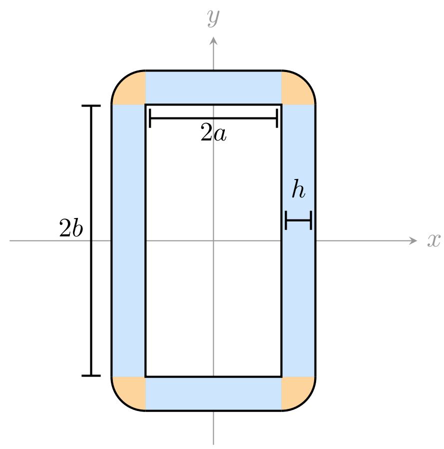
Thus, all three sides of the both triangles are the same, and we have congruence.

There are a number of interesting properties you can prove for these triangles via a number of other proof methods (for example, the exterior sides of our original triangle become the medians of the smaller triangles). Give it a try!

Extra Credit: This question explores some ideas that are going to come up waaaaaaaay at the end of the course, but we can still use a bit of basic geometry to get the right geometric idea in a few special cases.

(a) Let C_1 be the rectangle with vertices $(1, 2)$, $(-1, 2)$, $(-1, -2)$, and $(1, -2)$, and let C_2 be the "rectangle" with rounded corners that is always a distance h away from C_1 (see figure below). Find the area between C_1 and C_2. *Note:* The rounded corners of C_2 are circular arcs that are a distance h away from the nearest corner of C_1.

A possible solution: We can actually show this result in general, rather than just for the given rectangle. Suppose the given rectangle has upper-right corner at the coordinate (a, b). (In the problem statement we have $a = 1$, $b = 2$.) Let's begin by re-drawing our figure, this time taking care to distinguish between the circular parts and the non-circular parts.



The area is then broken up into four rectangular pieces and four circular pieces.

Two of the rectangular pieces have area $2b \cdot h$, and the other two rectangular pieces have area $2a \cdot h$. Thus, the total area of the rectangular (blue) bits is given by

$$2bh \cdot 2 + 2ah \cdot 2 = h\,(4a + 4b) = h \cdot (\text{perimeter of } C_1)$$

The circular (orange) bits each form a quarter of a circle of radius h, and there are 4 of them. As such, all of the orange pieces taken together make a full circle of radius h, which will have area

$$\pi h^2$$

The area between the two curves can then be found as

$$h \cdot (\text{perimeter of } C_1) + \pi h^2$$

For the specific rectangle given in the problem statement, we have $12h + \pi h^2$.

(b) Write the length of C_2 (the "perimeter" of the rounded rectangle) as the length of C_1 plus a corrective term. Explain how you find the corrective term.

A possible solution: More or less the same argument we applied in the last part will work here as well. Referring to the same figure, we see that the perimeter of C_2 is composed of 4 straight line segments and 4 circular arcs. There are two line segments of length $2a$ and two line segments of length $2b$. Each of the circular arcs is a quarter circle, and so all 4 arcs together will give a full circle, which will have perimeter $2\pi h$. Thus, the length of C_2 will be

$$(\text{perimeter of } C_1) + 2\pi h$$

This looks suspiciously related to our answer from part (a).

(c) Repeat the previous two parts for a new pair of curves, where C_1 is the circle of radius R centered at the origin, and C_2 is the circle of radius $R+h$ centered at the origin.

A possible solution: Let's begin with the areas. To find the area between two circles, we simply take the area of the larger circle and subtract the area of the smaller circle:

$$\pi (R+h)^2 - \pi R^2 = 2\pi Rh + \pi h^2 = h \cdot (\text{circumference of } C_1) + \pi h^2$$

Notice that this follows the same pattern we had in part (a): a term involving the length of the smaller curve, and a corrective term due to the circle.

For the circumference of the larger circle, we simply have

$$2\pi (R+h) = 2\pi R + 2\pi h = (\text{circumference of } C_1) + 2\pi h$$

Again, this result is the same as what we found in part (b).

(d) Do you think your answer to the previous part generalizes if we allow C_1 to be an arbitrary, but still "nice", planar curve, and we keep the restriction that C_2 can be formed by tracing out a curve around C_1 which is a constant width h away? What might you expect to find?

A possible solution: Based on our answers to the previous two parts, it is reasonable to think that the length of C_2 will be

$$(\text{length of } C_1) + 2\pi h$$

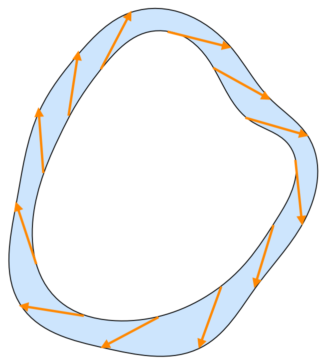
and the area between the two curves will be

$$h \cdot (\text{length of } C_1) + \pi h^2$$

When we get to chapter 16, we'll be able to show this result generically!

(e) Let's now turn to slightly different, but related question. For this problem we should note that a curve is *closed* if it begins where it ends, and a curve is *simple* if it does not intersect itself. Suppose you are riding a bike of body-length b, and you follow a path such that the front wheel of the bike traces out a simple, closed curve. (Nothing too wiggly – the rear wheel of the bike never intersects the path of the front wheel of the bike.) Necessarily, the rear wheel of the bike also traces out a related, but distinct, simple closed curve. Show that the area between these two curves is a constant that does not depend on the length of the curves. How does this relate to your answers to the previous parts? Intuitive (i.e. non-technical) explanations are welcome.

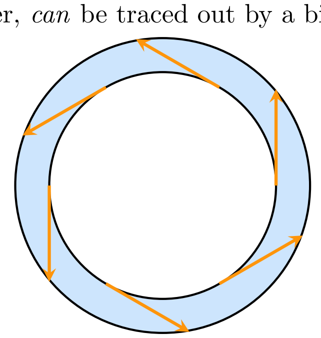
A possible solution: This question is so much fun I almost don't want to give you the answer! We'll be able to prove the relevant result for this problem directly once we've learned some of the material from chapter 16, but this problem actually has a very intuitive answer. Below, we sketch a cartoon of what the path of the bicycle must look like:



Notice that the shortest distance between the inner and outer curves is not necessarily a constant, like in the previous parts. There is, however, an additional constraint built into the problem: because the outer curve is constructed by following the path of the front wheel of the bike, it is always the case that for every point on the inner path (C_1), there is a line of length b connecting it to an associated point on the outer path (C_2). These line segments are marked in orange in the picture above. As we follow around the inner curve, these line segments of length b complete one full rotation. If we take each of those line segments and redraw them emanating from the same point, we get a circle of radius b. As such, the area between the two curves must be πb^2.

How does this square with our work from the first few parts? Well, the curves from parts (a) and (b) can't be traced out by a bike (at least not without some significant mechanical alterations to the bike), so this method shouldn't apply here.

The curves from part (c), however, *can* be traced out by a bike moving smoothly:



We need the direction of the front wheel of the bike to be tangent to the outer curve while the back wheel is tangent to the inner curve. The angle of the front wheel of the bicycle will determine how wide the strip between the circles is, and thus will determine h.