MATH 215 WINTER 2026
Homework Set 1: §12.2 - 12.5

Only some of the questions on this and other homework sets will be graded.
Due January 20, no later than 11:59pm, submitted through Gradescope.

You may work on these problems in groups (in fact, this is encouraged!), *but you must submit your own set of solutions.* Please neatly show your work!

Question 1: Suppose $\mathbf{v}$ and $\mathbf{u}$ are nonzero vectors. In class, we defined the *projection* of $\mathbf{v}$ onto $\mathbf{u}$. We define the *orthogonal projection* as the part of $\mathbf{v}$ that does *not* lie along $\mathbf{u}$:

$$\text{orth}_{\mathbf{u}}\mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v},$$

(a) Using the general formula for $\text{proj}_{\mathbf{u}}\mathbf{v}$, verify that $\text{orth}_{\mathbf{u}}\mathbf{v}$ is orthogonal to $\mathbf{u}$. (For this reason, we say that $\mathbf{v} = \text{orth}_{\mathbf{u}}\mathbf{v} + \text{proj}_{\mathbf{u}}\mathbf{v}$ is an *orthogonal decomposition* of $\mathbf{v}$.)

(b) In $\mathbb{R}^2$, compute the vectors $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{orth}_{\mathbf{u}}\mathbf{v}$ for $\mathbf{u} = \langle 1, -2\rangle$ and $\mathbf{v} = \langle 3, 1\rangle$. Sketch and clearly label $\mathbf{u}$, $\mathbf{v}$, $\text{proj}_{\mathbf{u}}\mathbf{v}$, and $\text{orth}_{\mathbf{u}}\mathbf{v}$.

(c) Compute $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{orth}_{\mathbf{u}}\mathbf{v}$ for $\mathbf{u} = \langle 2, -6, -3\rangle$ and $\mathbf{v} = \langle 8, 4, -6\rangle$.

Question 2:

(a) Describe all vectors in $\mathbb{R}^2$ orthogonal to $\mathbf{v} = \langle 3, -4\rangle$.

(b) Describe all vectors in $\mathbb{R}^3$ orthogonal to both $\mathbf{v} = \langle 1, 4, 2\rangle$ and $\mathbf{u} = \langle 2, 0, 1\rangle$.

(c) Find a vector $\mathbf{v}$ such that $\mathbf{v} \times \langle 5, -2, 6\rangle = \langle 2, 2, -1\rangle$. Is such a vector unique?

(d) Show that there is no vector $\mathbf{v}$ such that $\mathbf{v} \times \langle 5, -2, 6\rangle = \langle 1, -1, -1\rangle$.

(e) Describe all vectors in $\mathbb{R}^4$ orthogonal to $\mathbf{v} = \langle 1, 0, -1, 0\rangle$ and $\mathbf{u} = \langle 1, -1, 0, 1\rangle$. *Hint:* The cross product will not help you with this part.

Question 3: Consider the lines

$$\mathbf{r}_1(t) = \langle 1, -1, 2\rangle + t\,\langle -1, 4, 3\rangle \qquad \text{and} \qquad \mathbf{r}_2(s) = \langle -1, -3, 1\rangle + s\,\langle 1, 4, -2\rangle$$

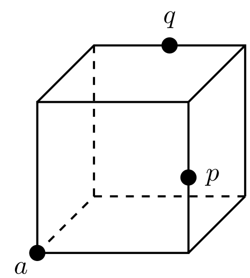
(a) Show that the lines are skew, that is, they are not parallel and they do not intersect (thus they are not contained in a plane).

(b) Find the points P on $\mathbf{r}_1$ and Q on $\mathbf{r}_2$ such that the distance $|PQ|$ is minimal.

(c) Find the equations of all planes parallel to the plane $8x + 3y - 5z = 2$ and 3 units away from it. *Hint:* Think about what it means for two planes to be parallel, and how to find the distance between two planes.

Question 4: Consider the planes P_1, P_2, and P_3, each with normal vector $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$, respectively. Assume that none of the planes are parallel. Let ℓ_{ij} be the line of intersection of the plane P_i with P_j. Is it possible for ℓ_{12}, ℓ_{23}, and ℓ_{13} to form a triangle? (Put another way, each pair of lines intersects precisely once.) If so, provide an example. If not, provide a convincing argument.

Question 5: Consider the cube pictured below:



The sides of the cube have length six. The points p and q are at the midpoints of their respective edges. Let T be the triangle with vertices a, p, and q.

(a) What is the area of T?

(b) If θ is the smallest angle of T, what is $\cos\theta$?

(c) The triangle T lives in a plane. What angle does this plane make with the bottom face of the cube?

Question 6: In section 12.5 of the book, we discovered that a line can generically be written in the form

$$\ell(t) = \langle x_0,\, y_0,\, z_0\rangle + t\mathbf{v},$$

where the point (x_0, y_0, z_0) is a known point on the line and $\mathbf{v}$ is the direction vector of the line. In this way, we can view t as your position on the line: in units of $|\mathbf{v}|$, it measures your distance from a known reference point.

A similar statement can be made for planes: Given any plane containing the point (x_0, y_0, z_0), every other point on the plane can be written as

$$\langle x_0, y_0, z_0\rangle + t\mathbf{v}_1 + s\mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are two non-parallel vectors that lie in the plane, and s and t are free parameters. (One might even say that s and t are a set of *coordinates* describing our position in the plane. This is what we mean when we say a plane is a two dimensional object–we need two parameters to navigate the space.)

Let's make this concrete. Consider the plane $x+y-4z = 2$, and take as our reference point $(x_0, y_0, z_0) = (1, 1, 0)$. To make the calculation easier, we will take $\mathbf{v}_1$ and $\mathbf{v}_2$ to be orthogonal unit vectors.

(a) Let $\mathbf{v}_1 = \left\langle \frac{1}{\sqrt{2}},\, -\frac{1}{\sqrt{2}},\, 0\right\rangle$. Find a $\mathbf{v}_2$ such that the following four conditions hold: (i) $|\mathbf{v}_2| = 1$, (ii) $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, (iii) $\mathbf{v}_2$ lies in the plane $x + y - 4z = 2$, and (iv) all components of $\mathbf{v}_2$ are positive.

(b) Starting at $(1, 1, 0)$, navigate to an arbitrary point (a, b, c) that lies on the plane moving only along the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Extra Credit: Let's apply the same basic idea from problem 1 to a slightly more general setting. Given an arbitrary set of vectors $\{\mathbf{v}_i\}_{i=1}^n$, we can construct a set of mutually orthogonal vectors from them, denoted $\{\mathbf{u}_i\}_{i=1}^n$. Let's illustrate how in the steps below.

(a) Begin with the vectors $\mathbf{v}_1 = \langle 1,\, 1,\, 1\rangle$, $\mathbf{v}_2 = \langle 2,\, 2,\, 4\rangle$ and $\mathbf{v}_3 = \langle -2,\, 2,\, -1\rangle$. Just so we have it for later on, find the plane containing $\mathbf{v}_1$, $\mathbf{v}_2$, and the point $(0, 0, 0)$.

(b) Let $\mathbf{u}_1 = \mathbf{v}_1$. Define $\mathbf{u}_2$ to be the orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$, and compute this quantity.

(c) We would now like to find a $\mathbf{u}_3$ such that $\mathbf{u}_3$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. For three dimensional vectors we have several ways of doing this, but let's try one specific way by trying to mimic the definition of the orthogonal projection, which will generalize to higher dimensions. Compute the projection of $\mathbf{v}_3$ onto $\mathbf{u}_1$ and the projection of $\mathbf{v}_3$ onto $\mathbf{u}_2$. Subtract both of these projections from $\mathbf{v}_3$, and denote the resulting vector by $\mathbf{u}_3$.

(d) Verify directly that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are all mutually pairwise orthogonal.

(e) Using your answer from part (a), can you find a relationship between the plane containing $\mathbf{v}_1$ and $\mathbf{v}_2$ and the vector $\mathbf{u}_3$? Explain.

(f) Now repeat and extend parts (b) through (d) for the following four vectors:

$$\mathbf{v}_1 = \langle 1, 0, 1, 0\rangle \qquad \mathbf{v}_2 = \langle -1, 1, 0, 0\rangle \qquad \mathbf{v}_3 = \langle 2, 0, 4, 1\rangle \qquad \mathbf{v}_4 = \langle 0, 3, 2, -4\rangle$$

At the end of this part you should have four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ with the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ if $i \neq j$.

(g) Why didn't we have you use the cross product in part (c) of this problem? Explain.