MATH 215 WINTER 2026
Homework Set 1: Minimal Solutions

Question 1: Suppose $\mathbf{v}$ and $\mathbf{u}$ are nonzero vectors. In class, we defined the *projection* of $\mathbf{v}$ onto $\mathbf{u}$. We define the *orthogonal projection* as the part of $\mathbf{v}$ that does *not* lie along $\mathbf{u}$:

$$\text{orth}_{\mathbf{u}}\mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v},$$

(a) Using the general formula for $\text{proj}_{\mathbf{u}}\mathbf{v}$, verify that $\text{orth}_{\mathbf{u}}\mathbf{v}$ is orthogonal to $\mathbf{u}$. (For this reason, we say that $\mathbf{v} = \text{orth}_{\mathbf{u}}\mathbf{v} + \text{proj}_{\mathbf{u}}\mathbf{v}$ is an *orthogonal decomposition* of $\mathbf{v}$.)

Solution: We will do this by computing the dot product of $\mathbf{u}$ with $\text{orth}_{\mathbf{u}}\mathbf{v}$. By definition we have

$$\mathbf{u} \cdot \text{orth}_{\mathbf{u}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \text{proj}_{\mathbf{u}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \left(\left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} \right) \mathbf{u} \right) = \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} \right) (\mathbf{u} \cdot \mathbf{u})$$

Recognizing that $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ allows us to cancel the denominator in the final term, giving

$$\mathbf{u} \cdot \text{orth}_{\mathbf{u}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$$

As the dot product is 0, we have shown that these two vectors are orthogonal.

(b) In $\mathbb{R}^2$, compute the vectors $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{orth}_{\mathbf{u}}\mathbf{v}$ for $\mathbf{u} = \langle 1, -2 \rangle$ and $\mathbf{v} = \langle 3, 1 \rangle$. Sketch and clearly label $\mathbf{u}$, $\mathbf{v}$, $\text{proj}_{\mathbf{u}}\mathbf{v}$, and $\text{orth}_{\mathbf{u}}\mathbf{v}$.

Solution: First, let's collect some useful information:

$$|\mathbf{u}|^2 = 5, \qquad |\mathbf{v}|^2 = 10, \qquad \mathbf{u} \cdot \mathbf{v} = 1$$

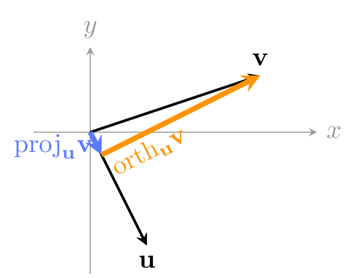
Using our definitions, we have

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} \right) \mathbf{u} = \frac{1}{5} \langle 1, -2 \rangle = \left\langle \frac{1}{5}, -\frac{2}{5} \right\rangle$$

and

$$\text{orth}_{\mathbf{u}}\mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v} = \langle 3, 1 \rangle - \left\langle \frac{1}{5}, -\frac{2}{5} \right\rangle = \left\langle \frac{14}{5}, \frac{7}{5} \right\rangle$$

Notice that, by construction, these two vectors are orthogonal to each other. A plot of the relevant vectors appears below:



(c) Compute $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{orth}_{\mathbf{u}}\mathbf{v}$ for $\mathbf{u} = \langle 2,\, -6,\, -3\rangle$ and $\mathbf{v} = \langle 8,\, 4,\, -6\rangle$.

Solution: Again, we collect some useful information:

$$|\mathbf{u}|^2 = 49, \qquad |\mathbf{v}|^2 = 116, \qquad \mathbf{u}\cdot\mathbf{v} = 10$$

And again, we apply our definition to find

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|^2}\right)\mathbf{u} = \frac{10}{49}\langle 2,\, -6,\, -3\rangle = \left\langle \frac{20}{49},\, -\frac{60}{49},\, -\frac{30}{49}\right\rangle$$

and

$$\text{orth}_{\mathbf{u}}\mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v} = \langle 8,\, 4,\, -6\rangle - \left\langle \frac{20}{49},\, -\frac{60}{49},\, -\frac{30}{49}\right\rangle = \left\langle \frac{372}{49},\, \frac{256}{49},\, -\frac{264}{49}\right\rangle$$

You can check for yourself that these two vectors are orthogonal, as expected. You can even plot them yourself (e.g. using Desmos) and see that they make a picture very similar to what you drew for part (b).

Question 2:

(a) Describe all vectors in $\mathbb{R}^2$ orthogonal to $\mathbf{v} = \langle 3,\, -4\rangle$.

Solution: Suppose we have a vector $\mathbf{u} = \langle a,\, b\rangle$. This vector is orthogonal to $\mathbf{v}$ if and only if the dot product is zero. We find

$$\mathbf{u}\cdot\mathbf{v} = 3a - 4b = 0$$

Thus, every such vector lies on a line satisfying $3a - 4b = 0$. That is, every such $\mathbf{u}$ is parallel to the vector $\langle 4,\, 3\rangle$, and so we might say $\mathbf{u} = \alpha\langle 4,\, 3\rangle$ for $\alpha \in \mathbb{R}$.

(b) Describe all vectors in $\mathbb{R}^3$ orthogonal to both $\mathbf{v} = \langle 1,\, 4,\, 2\rangle$ and $\mathbf{u} = \langle 2,\, 0,\, 1\rangle$.

Solution: Here we can use the fact that the cross product is orthogonal to both vectors used to generate it. Because we are in $\mathbb{R}^3$, there will be a unique direction (but not a unique vector) that is orthogonal to both $\mathbf{v}$ and $\mathbf{u}$. We see

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 4 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix}\hat{k} = 4\hat{\imath} + 3\hat{\jmath} - 8\hat{k} = \langle 4,\, 3,\, -8\rangle$$

and thus any vector parallel to this vector will be orthogonal to both $\mathbf{v}$ and $\mathbf{u}$. We might reasonably say that any vector $\mathbf{w}$ that is orthogonal to $\mathbf{v}$ and $\mathbf{u}$ will be of the form

$$\mathbf{w} = \alpha\langle 4,\, 3,\, -8\rangle \qquad \text{for some } \alpha \in \mathbb{R}$$

You can also solve this problem by supposing $\mathbf{w} = \langle a,\, b,\, c\rangle$ and forcing $\mathbf{w}\cdot\mathbf{v} = 0$ and $\mathbf{w}\cdot\mathbf{u} = 0$, but that will (in principle) take considerably more time and algebra.

(c) Find a vector $\mathbf{v}$ such that $\mathbf{v}\times\langle 5,\, -2,\, 6\rangle = \langle 2,\, 2,\, -1\rangle$. Is such a vector unique?

Solution: There are a number of ways to do this problem. If all we need to do is find *one* such example, we can actually apply more or less the same idea as last time, and simply compute $\langle 2,\, 2,\, -1\rangle \times \langle 5,\, -2,\, 6\rangle$ to find a potential direction for $\mathbf{v}$, and then renormalize the result to give us the vector we want. That process should yield the upsetting answer of

$$\mathbf{v} = \left\langle -\frac{30}{195}, \frac{51}{195}, \frac{42}{195} \right\rangle$$

If we want to say something slightly more general, let $\mathbf{v} = \langle a,\, b,\, c\rangle$. The cross product gives

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ 5 & -2 & 6 \end{vmatrix} = (6b + 2c)\,\hat{\imath} + (5c - 6a)\,\hat{\jmath} - (2a + 5b)\,\hat{k}$$

Forcing equality of the desired quantities, we arrive at the set of equations

$$3b + c = 1 \qquad 5c - 6a = 2 \qquad 2a + 5b = 1$$

Solving this system of equations for, say, a and c in terms of b, we will find

$$a = \frac{1 - 5b}{2} \qquad c = 1 - 3b$$

This shows that there is a whole line of potential solutions. One such simple solution is $a = 1/2$, $b = 0$, $c = 1$. Another simple solution is $a = -2$, $b = 1$, $c = -2$. You can check that the solution we found earlier also lies on this line.

(d) Show that there is no vector $\mathbf{v}$ such that $\mathbf{v} \times \langle 5,\, -2,\, 6\rangle = \langle 1,\, -1,\, -1\rangle$.

Solution: Recall that the vector resulting from a cross product will be orthogonal to *both* vectors used in the cross product. Thus, if such a vector $\mathbf{v}$ exists, we must have

$$\mathbf{v} \cdot \langle 1,\, -1,\, -1\rangle = 0 \qquad \text{and} \qquad \langle 5,\, -2,\, 6\rangle \cdot \langle 1,\, -1,\, -1\rangle = 0$$

However, a direct computation shows that

$$\langle 5,\, -2,\, 6\rangle \cdot \langle 1,\, -1,\, -1\rangle = 5 + 2 - 6 = 1 \neq 0$$

Thus, no such $\mathbf{v}$ exists.

(e) Describe all vectors in $\mathbb{R}^4$ orthogonal to $\mathbf{v} = \langle 1, 0, -1, 0\rangle$ and $\mathbf{u} = \langle 1, -1, 0, 1\rangle$. *Hint:* The cross product will not help you with this part.

Solution: Suppose our arbitrary vector is given by $\langle a,\, b,\, c,\, d\rangle$. Then forcing this to be orthogonal to both $\mathbf{v}$ and $\mathbf{u}$ gives

$$\langle a,\, b,\, c,\, d\rangle \cdot \mathbf{v} = a - c = 0 \qquad \text{and} \qquad \langle a,\, b,\, c,\, d\rangle \cdot \mathbf{u} = a - b + d = 0$$

So we need $c = a$ and $b = a + d$. Notice that this requires *two* constants to uniquely specify. This means we will have a plane (or more specifically something called a *hyperplane*). That is, if we specify a and d, then we get c and b for free, but if we fail to specify d then we have many answers for b. Suppose we specify a and a. Then we have $\langle a,\, a + d,\, a,\, d\rangle$.
To make the planar structure explicit, we can find any two nonparallel vectors that follow the above pattern. For example, one such set of choices might be

$$\mathbf{w}_1 = \langle 1,\, 1,\, 1,\, 0\rangle \qquad \text{and} \qquad \mathbf{w}_2 = \langle 0,\, 1,\, 0,\, 1\rangle$$

Then any vector of the form $\langle a,\, a + d,\, a,\, d\rangle$ can be written as the linear combination $a\mathbf{w}_1 + d\mathbf{w}_2$.

Question 3: Consider the lines

$$\mathbf{r}_1(t) = \langle 1,\, -1,\, 2\rangle + t\,\langle -1,\, 4,\, 3\rangle \qquad \text{and} \qquad \mathbf{r}_2(s) = \langle -1,\, -3,\, 1\rangle + s\,\langle 1,\, 4,\, -2\rangle$$

(a) Show that the lines are skew, that is, they are not parallel and they do not intersect (thus they are not contained in a plane).

Solution: Each line is of the form $\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}$, where $\mathbf{p}$ is the *starting point* and $\mathbf{v}$ is the *direction vector*. If two lines are parallel, their direction vectors will be parallel. Our direction vectors are $\langle -1,\, 4,\, 3\rangle$ and $\langle 1,\, 4,\, -2\rangle$. I leave it to you to show that these two vectors are not parallel (i.e. show there is no possible number c such that $\langle -1,\, 4,\, 3\rangle = c\,\langle 1,\, 4,\, -2\rangle$).

Next, we want to show that the lines do not intersect. If the lines intersect, then there must be a value of t and a value of s such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$. This condition is equivalent to the following set of equations:

$$1 - t = s - 1, \qquad 4t - 1 = 4s - 3, \qquad 2 + 3t = 1 - 2s$$

Solving, say, the left and center equations simultaneously, we arrive at an answer of

$$t = \frac{3}{4} \qquad s = \frac{5}{4}$$

However, if we put these values into the rightmost equation, we find

$$2 + 3\left(\frac{3}{4}\right) = 1 - 2\left(\frac{5}{4}\right) \qquad \Rightarrow \qquad 2 + \frac{9}{4} + 3 = 1 - \frac{5}{2} \qquad \Rightarrow \qquad \frac{17}{4} = -\frac{3}{2}$$

As these two numbers are not actually equal, there is no pair of values for t and s at which the lines occupy the same point in space. Thus the lines do not intersect.

Because the lines do not intersect and are not parallel, we say they are *skew.*

(b) Find the points P on $\mathbf{r}_1$ and Q on $\mathbf{r}_2$ such that the distance $|PQ|$ is minimal.

Solution: There are a number of ways to solve this problem–here we will employ a geometric approach, but we could revisit this question after we start our work in chapter 14!

For this problem, we are going to use the fact that the distance will be minimal when the vector between the two positions is orthogonal to both lines. That is, we consider each line to be

$$\mathbf{r}_1(t) = \mathbf{p}_1 + t\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{r}_2(s) = \mathbf{p}_2 + s\mathbf{v}_2$$

and we require that the difference in positions is orthogonal to both direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, we have

$$(\mathbf{r}_1(t) - \mathbf{r}_2(s)) \cdot \mathbf{v}_1 = 0 \qquad \text{and} \qquad (\mathbf{r}_1(t) - \mathbf{r}_2(s)) \cdot \mathbf{v}_2 = 0$$

This gives

$$\mathbf{r}_1(t) - \mathbf{r}_2(s) = \langle 2 - t - s,\, 2 + 4t - 4s,\, 1 + 3t + 2s\rangle$$

The first dot product gives

$$(\mathbf{r}_1(t) - \mathbf{r}_2(s)) \cdot \mathbf{v}_1 = 9 + 26t - 9s = 0$$

while the second dot product gives

$$(\mathbf{r}_1(t) - \mathbf{r}_2(s)) \cdot \mathbf{v}_2 = 8 + 9t - 21s = 0$$

There is some algebra required to solve this system of equations, the result of which will give

$$t = -\frac{39}{155} \qquad s = \frac{127}{465}$$

Because the answer asked for the points P and Q, we have

$$P = \mathbf{r}_1(t) = \frac{1}{155}\langle 194,\, -311,\, 193\rangle$$

$$Q = \mathbf{r}_2(s) = \frac{1}{465}\langle -338,\, -887,\, 211\rangle$$

(c) Find the equations of all planes parallel to the plane $8x + 3y - 5z = 2$ and 3 units away from it. *Hint:* Think about what it means for two planes to be parallel, and how to find the distance between two planes.

Solution: Since we are looking for parallel planes, all we need to do is find a point on the given plane and move along the direction of the normal vector the appropriate distance to a new point. We will now have a point and a normal vector, and can thus find the equation of our plane. You can find your favorite point on the given plane, but I quite like $p = (0, -1, -1)$. Next, we see that the normal vector is given by $\mathbf{n} = \langle 8, 3, -5\rangle$, which gives a unit normal vector of

$$\hat{n} = \left\langle \frac{8}{\sqrt{98}}, \frac{3}{\sqrt{98}}, -\frac{5}{\sqrt{98}} \right\rangle$$

To find our new points, we want to move "up" and "down" the direction of the normal vector a distance of 4. This will put us at the points

$$q_1 = p + 3\hat{n} = \left\langle \frac{24}{\sqrt{98}}, -1 + \frac{9}{\sqrt{98}}, -1 - \frac{15}{\sqrt{98}} \right\rangle$$

and

$$q_2 = p - 3\hat{n} = \left\langle -\frac{24}{\sqrt{98}}, -1 - \frac{9}{\sqrt{98}}, -1 + \frac{15}{\sqrt{98}} \right\rangle$$

Using the point q_1, our first plane is found by

$$8\left(x - \frac{24}{\sqrt{98}}\right) + 3\left(y + 1 - \frac{9}{\sqrt{98}}\right) - 5\left(z + 1 + \frac{15}{\sqrt{98}}\right) = 0 \qquad \Rightarrow \qquad 8x + 3y - 5z = 2 + \frac{294}{\sqrt{98}}$$

Using the point q_2, our second plane is found by

$$8\left(x + \frac{24}{\sqrt{98}}\right) + 3\left(y + 1 + \frac{9}{\sqrt{98}}\right) - 5\left(z + 1 - \frac{15}{\sqrt{98}}\right) = 0 \qquad \Rightarrow \qquad 8x + 3y - 5z = 2 - \frac{294}{\sqrt{98}}$$

Question 4: Consider the planes P_1, P_2, and P_3, each with normal vector $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$, respectively. Assume that none of the planes are parallel. Let ℓ_{ij} be the line of intersection of the plane P_i with P_j. Is it possible for ℓ_{12}, ℓ_{23}, and ℓ_{13} to form a triangle? (Put another way, each pair of lines intersects precisely once.) If so, provide an example. If not, provide a convincing argument.

Solution: It is not possible for a triangle to be formed by three such planes. There are many ways you can show this–we present only one below.

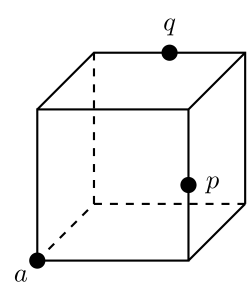
First, let's suppose that ℓ_{12} and ℓ_{23} intersect at a point p. Because ℓ_{12} lies in both planes P_1 and P_2, we must have p lies in both planes. Similarly, because ℓ_{23} lies in both planes P_2 and P_3, we must also have that p lies in plane P_3. Thus, any intersection of lines ℓ_{12} and ℓ_{23} lies in all three planes P_1, P_2, and P_3.

An entirely analogous argument will show that any intersection of ℓ_{12} and ℓ_{13} will lie in all three planes, as will any intersection of ℓ_{23} and ℓ_{13}.

Let us now suppose that we have a triangle formed by these three lines. Each vertex of this triangle is an intersection of precisely two of the lines, and therefore each vertex of the triangle lies in all three planes P_1, P_2, and P_3. However, because three points determine a plane in three dimensions*, this means that P_1, P_2, and P_3 must coincide. Because the problems statement specifies that none of the planes are parallel, we have arrived at a contradiction, and thus it is impossible to form such a triangle.

* We could also make this argument slightly more rigorous using various facts about cross products. Once we have that all vertices of the triangle lie in all three planes, we get for free that all three lines lie in all three planes, and we can now use the fact that the cross product uniquely defines the normal vector of a plane to arrive at the same contradiction.

Question 5: Consider the cube pictured below:



The sides of the cube have length six. The points p and q are at the midpoints of their respective edges. Let T be the triangle with vertices a, p, and q.

Solution: A question similar to this appeared on first midterm from fall 2019. This problem becomes much easier if we put a coordinate system down. The choice here doesn't really matter too much, but it will be convenient to let the point a be at the origin. With this choice, we consider each vector pointing from the origin to the point, giving

$$\vec{a} = \langle 0,\, 0,\, 0 \rangle \qquad \vec{p} = \langle 6,\, 0,\, 3 \rangle \qquad \vec{q} = \langle 3,\, 6,\, 6 \rangle$$

In this coordinate system the bottom front edge of the cube is the positive x-axis, the left front edge of the cube is the positive z-axis, and the left dashed line is the positive y-axis.

(a) What is the area of T?

Solution: The triangle T is half the parallelogram formed by the vectors $\vec{p}$ and $\vec{q}$. Thus, the area of T will be

$$\text{Area}(T) = \frac{1}{2}\left|\vec{p} \times \vec{q}\right|$$

Let's first find the cross product:

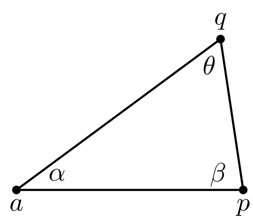
$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 0 & 3 \\ 3 & 6 & 6 \end{vmatrix} = \langle -18,\, -27,\, 36\rangle = -9\,\langle 2,\, 3,\, -4\rangle$$

The area of the triangle is then

$$\text{Area}(T) = \frac{1}{2}\left|\vec{p} \times \vec{q}\right| = \frac{1}{2} \cdot 9 \cdot \left|\langle 2,\, 3,\, -4\rangle\right| = \frac{9}{2}\sqrt{4+9+16} = \frac{9}{2}\sqrt{29}$$

(b) If θ is the smallest angle of T, what is $\cos\theta$?

Solution: From the picture I think it is not clear which of the three angles is the smallest. Let's just find all of them and decide which one is theta. If we look down on our triangle from above, we have a picture that looks roughly like this:



Visually, it is quite hard to decide which angle is the smallest. Because each of these angles lies between 0 and $\pi/2$, we know that the smallest angle value will have the largest cosine value. So let's find the cosine of all three angles and decide which is the smallest.

For the angle marked α, we want the vectors that point from a to p and from a to q. There are simply the vectors $\vec{p}$ and $\vec{q}$ we gave in the preamble. We find

$$\cos\alpha = \frac{\vec{p}\cdot\vec{q}}{\left|\vec{p}\right|\left|\vec{q}\right|} = \frac{36}{9\sqrt{45}} = \frac{4}{\sqrt{45}}$$

For the angle marked β, we want the vectors that point from p to a and from p to q. Respectively, this gives

$$\overrightarrow{pa} = -\vec{p} = \langle -6,\, 0,\, -3\rangle \qquad \overrightarrow{pq} = \vec{q} - \vec{p} = \langle -3,\, 6,\, 3\rangle$$

This will give

$$\overrightarrow{pa}\cdot\overrightarrow{pq} = 9 \qquad \left|\overrightarrow{pa}\right| = \sqrt{45} \qquad \left|\overrightarrow{pq}\right| = \sqrt{54}$$

which gives

$$\cos\beta = \frac{\overrightarrow{pa}\cdot\overrightarrow{pq}}{\left|\overrightarrow{pa}\right|\left|\overrightarrow{pq}\right|} = \frac{9}{\sqrt{45}\cdot\sqrt{54}} = \frac{1}{\sqrt{30}}$$

For the angle marked θ, we want the vectors that point from q to a and from q to p. Respectively, this gives

$$\overrightarrow{qa} = -\vec{q} = \langle -3,\, -6,\, -6\rangle \qquad \overrightarrow{qp} = \vec{p} - \vec{q} = \langle 3,\, -6,\, -3\rangle$$

This will give

$$\overrightarrow{qa}\cdot\overrightarrow{qp} = 45 \qquad \left|\overrightarrow{qa}\right| = 9 \qquad \left|\overrightarrow{qp}\right| = \sqrt{54}$$

which gives

$$\cos\theta = \frac{\overrightarrow{qa}\cdot\overrightarrow{qp}}{|\overrightarrow{qa}|\,|\overrightarrow{qp}|} = \frac{45}{9\cdot\sqrt{54}} = \frac{5}{\sqrt{54}}$$

You can pick your favorite method to verify that

$$\frac{1}{\sqrt{30}} < \frac{4}{\sqrt{45}} < \frac{5}{\sqrt{54}},$$

so we can see that the angle we marked θ is actually the smallest.

(c) The triangle T lives in a plane. What angle does this plane make with the bottom face of the cube?

Solution: In part (a) we found that $\vec{p}\times\vec{q}$ is parallel to the vector $\langle -2,\,-3,\,4\rangle$. Taking this as the normal vector to the plane of the triangle, and recognizing that the triangle contains the origin, we know the triangle T lies in the plane $2x+3y-4z=0$. The bottom face of the cube lies in the xy-plane in this coordinate system. The angle between the triangle and the bottom face of the cube is now the angle between the plane $2x+3y-4z=0$ and the plane $z=0$. The angle between two planes is the same as the angle between their normal vectors. These normal vectors are $\mathbf{n}_1 = \langle -2,\,-3,\,4\rangle$ and $\mathbf{n}_2 = \langle 0,\,0,\,1\rangle$.

Let γ be the angle we want. Then γ satisfies

$$\cos\gamma = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{4}{\sqrt{29}} \qquad \Rightarrow \qquad \gamma = \cos^{-1}\left(\frac{4}{\sqrt{29}}\right) \approx 0.733\,\text{radians} \approx 42.03^\circ$$

Question 6: In section 12.5 of the book, we discovered that a line can generically be written in the form

$$\ell(t) = \langle x_0,\,y_0,\,z_0\rangle + t\mathbf{v},$$

where the point (x_0, y_0, z_0) is a known point on the line and $\mathbf{v}$ is the direction vector of the line. In this way, we can view t as your position on the line: in units of $|\mathbf{v}|$, it measures your distance from a known reference point.

A similar statement can be made for planes: Given any plane containing the point (x_0, y_0, z_0), every other point on the plane can be written as

$$\langle x_0, y_0, z_0\rangle + t\mathbf{v}_1 + s\mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are two non-parallel vectors that lie in the plane, and s and t are free parameters. (One might even say that s and t are a set of *coordinates* describing our position in the plane. This is what we mean when we say a plane is a two dimensional object–we need two parameters to navigate the space.)

Let's make this concrete. Consider the plane $x+y-4z=2$, and take as our reference point $(x_0, y_0, z_0) = (1, 1, 0)$. To make the calculation easier, we will take $\mathbf{v}_1$ and $\mathbf{v}_2$ to be orthogonal unit vectors.

(a) Let $\mathbf{v}_1 = \left\langle \frac{1}{\sqrt{2}},\,-\frac{1}{\sqrt{2}},\,0\right\rangle$. Find a $\mathbf{v}_2$ such that the following four conditions hold: (i) $|\mathbf{v}_2| = 1$, (ii) $\mathbf{v}_1\cdot\mathbf{v}_2 = 0$, (iii) $\mathbf{v}_2$ lies in the plane $x+y-4z=2$, and (iv) all components of $\mathbf{v}_2$ are positive.

Solution: Because $\mathbf{v}_2$ lies in the plane $x+y-4z=2$, we know that $\mathbf{v}_2$ must be orthogonal to $\mathbf{n} = \langle 1,\,1,\,-4\rangle$. We can either solve for $\mathbf{v}_2$ by manually forcing both orthogonality conditions ($\mathbf{v}_1\cdot\mathbf{v}_2 = 0$ and $\mathbf{v}_2\cdot\mathbf{n} = 0$), or we can use the fact that in three dimensions we can use the cross product to help us here. The cross product requires significantly less algebra, so let's run with that one. We have that $\mathbf{v}_2$ must be parallel to $\mathbf{v}_1\times\mathbf{n}$. To make the algebra easier, let's compute $\sqrt{2}\mathbf{v}_2 \times \mathbf{n}$:

$$\sqrt{2}\mathbf{v}_1 \times \mathbf{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & -4 \end{vmatrix} = \langle 4,\, 4,\, 2 \rangle$$

This means that $\mathbf{v}_2$ must be parallel to the vector $\langle 2,\, 2,\, 1\rangle$. We can take $\mathbf{v}_2 = c\,\langle 2,\, 2,\, 1\rangle$, and recognize that $|\mathbf{v}_2| = 3\,|c|$, giving our choice of $\mathbf{v}_2$ as

$$\mathbf{v}_2 = \left\langle \frac{2}{3},\, \frac{2}{3},\, \frac{1}{3} \right\rangle$$

Our choice of c makes this vector have length 1, by construction it is orthogonal to the desired two vectors, and it self-evidently has all positive entries, so this meets all of our criteria.

(b) Starting at $(1, 1, 0)$, navigate to an arbitrary point (a, b, c) that lies on the plane moving only along the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: This question is asking us to solve the following equation for x and y:

$$\langle a,\, b,\, c\rangle = \langle 1,\, 1,\, 0\rangle + x\mathbf{v}_1 + y\mathbf{v}_2$$

We start at the point $(1, 1, 0)$, move x amount in the direction of $\mathbf{v}_1$ and y amount in the direction of $\mathbf{v}_2$, and we end up at the point (a, b, c). We are going to think of x and y as the *coordinates in the plane.* Writing the above equation out component wise, we have

$$\begin{aligned} a &= 1 + \frac{x}{\sqrt{2}} + \frac{2y}{3} \\ b &= 1 - \frac{x}{\sqrt{2}} + \frac{2y}{3} \\ c &= 0 - 0x + \frac{y}{3} \end{aligned}$$

You can verify for yourself that $a + b - 4c = 2$, so we know this point will actually live in the plane it is supposed to.

All we need do is solve for x and y in terms of a, b, and c. The third equation will give us $y = 3c$ simply enough, and if we subtract the second equation from the first we will find

$$a - b = \frac{2x}{\sqrt{2}} \qquad \Rightarrow \qquad x = \frac{a-b}{\sqrt{2}}$$

You should check your algebra to make sure that this answer works (it does, but you should check).

Extra Credit: Let's apply the same basic idea from problem 1 to a slightly more general setting. Given an arbitrary set of vectors $\{\mathbf{v}_i\}_{i=1}^n$, we can construct a set of mutually orthogonal vectors from them, denoted $\{\mathbf{u}_i\}_{i=1}^n$. Let's illustrate how in the steps below.

(a) Begin with the vectors $\mathbf{v}_1 = \langle 1,\, 1,\, 1\rangle$, $\mathbf{v}_2 = \langle 2,\, 2,\, 4\rangle$ and $\mathbf{v}_3 = \langle -2,\, 2,\, -1\rangle$. Just so we have it for later on, find the plane containing $\mathbf{v}_1$, $\mathbf{v}_2$, and the point $(0, 0, 0)$.

Solution: Because this plane contains $\mathbf{v}_1$ and $\mathbf{v}_2$, we can find the normal vector by taking the cross product. We have

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{vmatrix} = \langle 2,\, -2,\, 0 \rangle$$

This plane contains the point $(0,0,0)$, so the equation of the plane is given by

$$\mathbf{n}\cdot(\langle x,\, y,\, z\rangle - \langle 0,0,0\rangle) = 0 \qquad \Rightarrow \qquad 2x - 2y = 0 \qquad \Rightarrow \qquad x - y = 0$$

Thus, they live in the plane $x = y$.

(b) Let $\mathbf{u}_1 = \mathbf{v}_1$. Define $\mathbf{u}_2$ to be the orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$, and compute this quantity.

Solution: First, we will collect a few convenient quantities:

$$\mathbf{u}_1 = \langle 1,\, 1,\, 1\rangle, \qquad |\mathbf{u}_1| = \sqrt{3}, \qquad \mathbf{v}_2\cdot\mathbf{u}_1 = 8$$

Then we find the projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$ is

$$\text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{|\mathbf{u}_1|^2}\,\mathbf{u}_1 = \frac{8}{3}\mathbf{u}_1 = \left\langle \frac{8}{3},\, \frac{8}{3},\, \frac{8}{3}\right\rangle$$

The orthogonal projection is then found as

$$\mathbf{u}_2 = \text{orth}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{8}{3}\mathbf{u}_1 = \left\langle -\frac{2}{3},\, -\frac{2}{3},\, \frac{4}{3}\right\rangle$$

(c) We would now like to find a $\mathbf{u}_3$ such that $\mathbf{u}_3$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. For three dimensional vectors we have several ways of doing this, but let's try one specific way by trying to mimic the definition of the orthogonal projection, which will generalize to higher dimensions. Compute the projection of $\mathbf{v}_3$ onto $\mathbf{u}_1$ and the projection of $\mathbf{v}_3$ onto $\mathbf{u}_2$. Subtract both of these projections from $\mathbf{v}_3$, and denote the resulting vector by $\mathbf{u}_3$.

Solution: As before, it will be convenient to collect some convenient quantities that we're going to use in our computations. We have

$$\mathbf{u}_1 = \langle 1,\, 1,\, 1\rangle, \qquad |\mathbf{u}_1| = \sqrt{3}, \qquad \mathbf{v}_3\cdot\mathbf{u}_1 = -1$$

which gives

$$\text{proj}_{\mathbf{u}_1}\mathbf{v}_3 = \frac{\mathbf{v}_3\cdot\mathbf{u}_1}{|\mathbf{u}_1|^2}\,\mathbf{u}_1 = -\frac{1}{3}\mathbf{u}_1 = \left\langle -\frac{1}{3},\, -\frac{1}{3},\, -\frac{1}{3}\right\rangle$$

Similarly, we have

$$\mathbf{u}_2 = \frac{2}{3}\langle -1,\, -1,\, 2\rangle, \qquad |\mathbf{u}_2| = \sqrt{\frac{8}{3}}, \qquad \mathbf{v}_3\cdot\mathbf{u}_2 = -\frac{4}{3}$$

which will give the projection as

$$\text{proj}_{\mathbf{u}_2}\mathbf{v}_3 = \frac{\mathbf{v}_3\cdot\mathbf{u}_2}{|\mathbf{u}_2|^2}\,\mathbf{u}_2 = -\frac{1}{2}\mathbf{u}_2 = \left\langle \frac{1}{3},\, \frac{1}{3},\, -\frac{2}{3}\right\rangle$$

Our definition of $\mathbf{u}_3$ is then

$$\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{u}_2}\mathbf{v}_3 = \mathbf{v}_3 + \frac{1}{3}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 = \langle -2, 2, 0\rangle$$

(d) Verify directly that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are all mutually pairwise orthogonal.

Solution: This is straightforward enough:

$$\begin{aligned}
\mathbf{u_1} \cdot \mathbf{u}_2 &= \langle 1,\, 1,\, 1\rangle \cdot \frac{2}{3}\langle -1,\, -1, 2\rangle = \frac{2}{3}(-1-1+2) = 0 \\
\mathbf{u_1} \cdot \mathbf{u}_3 &= \langle 1,\, 1,\, 1\rangle \cdot \langle -2,\, 2,\, 0\rangle = -2+2+0 = 0 \\
\mathbf{u_2} \cdot \mathbf{u}_3 &= \frac{2}{3}\langle -1,\, -1, 2\rangle \cdot \langle -2,\, 2,\, 0\rangle = \frac{2}{3}(2-2+0) = 0
\end{aligned}$$

As each dot product is zero, each pair of vectors is orthogonal.

(e) Using your answer from part (a), can you find a relationship between the plane containing $\mathbf{v}_1$ and $\mathbf{v}_2$ and the vector $\mathbf{u}_3$? Explain.

Solution: The vector $\mathbf{u}_3$ is parallel to the normal vector of the plane containing $\mathbf{v}_1$ and $\mathbf{v}_2$. Because we are in three dimensions, this had to be the case: our algorithm above is designed to iteratively generate orthogonal vectors. Notice that, though $\mathbf{u}_2$ is not parallel to $\mathbf{v}_2$, it is the case that $\mathbf{u}_2$ lies in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, any vector that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ must also be orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, and in three dimensions that leaves only one option: the normal vector to the plane containing $\mathbf{v}_1$ and $\mathbf{v}_2$.

(f) Now repeat and extend parts (b) through (d) for the following four vectors:

$$\mathbf{v}_1 = \langle 1,\, 0,\, 1,\, 0\rangle \qquad \mathbf{v}_2 = \langle -1,\, 1,\, 0,\, 0\rangle \qquad \mathbf{v}_3 = \langle 2,\, 0,\, 4,\, 1\rangle \qquad \mathbf{v}_4 = \langle 0,\, 3,\, 2,\, -4\rangle$$

At the end of this part you should have four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ with the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ if $i \neq j$.

Solution: In this problem you are performing a variant of an algorithm called *Gram-Schmidt orthogonalization*, which will iteratively generate orthogonal vectors in any number of dimensions. Let's show the details here briefly–I will leave much of the algebra to you.

First, take $\mathbf{u}_1 = \mathbf{v}_1 = \langle 1,\, 0,\, 1,\, 0\rangle$. Because it will be convenient throughout the rest of the problem, we note $|\mathbf{u}_1| = \sqrt{2}$.

To find $\mathbf{u}_2$, we take

$$\mathbf{u}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{|\mathbf{u}_1|^2}\right)\mathbf{u}_1 = \mathbf{v}_2 + \frac{1}{2}\mathbf{u}_1 = \left\langle -\frac{1}{2},\, 1,\, \frac{1}{2},\, 0\right\rangle$$

Because we're going to want it, we note $|\mathbf{u}_2| = \sqrt{3/2}$.

To find $\mathbf{u}_3$, we take

$$\mathbf{u}_3 = \mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_2}\mathbf{v}_3$$

I'll leave the specific dot product calculations to you, but you should find

$$\mathbf{u}_3 = \mathbf{v}_3 - 3\mathbf{u}_1 - \frac{2}{3}\mathbf{u}_2 = \left\langle -\frac{2}{3},\, -\frac{2}{3},\, \frac{2}{3},\, 1\right\rangle$$

As has been the case, it will be helpful to write down the length of this vector, giving $|\mathbf{u}_3| = \sqrt{7/3}$.

Repeating this process one final time, we take

$$\mathbf{u}_4 = \mathbf{v}_4 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{v}_4 - \operatorname{proj}_{\mathbf{u}_2}\mathbf{v}_4 - \operatorname{proj}_{\mathbf{u}_3}\mathbf{v}_4$$

Leaving the algebra to you, you should find

$$\mathbf{u}_4 = \mathbf{v}_4 - \mathbf{u}_1 - \frac{8}{3}\mathbf{u}_2 + 2\mathbf{u}_3 = \langle -1,\ -1,\ 1,\ -2 \rangle$$

We leave it to you to verify that all four of these vectors are pairwise orthogonal.

(g) Why didn't we have you use the cross product in part (c) of this problem? Explain.

Solution: In part (c) you could have used the cross product to find a vector orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, but that will only work in three dimensions. In order to generalize this process to work in any number of dimensions, we need another way to computing orthogonal vectors, and we have demonstrated that here: subtract off all of the pieces of the vector that lie along a given vector (i.e. the projection), and all that remains will be orthogonal to the directions we subtracted off.