MATH 215 WINTER 2026
Homework Set 4: §14.1 - 14.6 (just barely)

Only some of the questions on this and other homework sets will be graded.
Due February 10, no later than 11:59pm, submitted through Gradescope.

You may work on these problems in groups (in fact, this is encouraged!), *but you must submit your own set of solutions*. Please neatly show your work! Submissions that show no work may receive no credt.

Question 1: Let $f(x, y) = xe^{-y^2} - ye^{-x^2}$.

(a) Find the equation for the tangent plane to the graph of f at the point $(2, 1)$.

(b) If one exists, find a point on the surface $z = x^2 - y^2$ has a tangent plane parallel to the plane found in the previous part. If one does not exist, justify why.

Question 2: A function of two variables $u = u(x, t)$ is said to satisfy the *wave equation* in one space dimension if it satisfies the identity $u_{tt} = c^2 u_{xx}$. Here $c > 0$ is a constant denoting the speed of propagation of the wave.

(a) Take f and g to be two twice-differentiable functions of one variable. Show that

$$u(x, t) = f(x - ct) + g(x + ct)$$

is a solution of the wave equation.

(b) One can show (but you don't have to) that all solutions of the one dimensional wave equation are of the above form for *some* f and g. Use this fact to find the solution of the wave equation that satisfies the initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = xe^{-x^2/2}$$

(c) Determine which, if any, of the following functions are solutions to Laplace's equation $u_{xx} + u_{yy} = 0$:

$$f(x, y) = \frac{y}{a^2y^2 - x^2} \qquad g(x, y) = e^{-x}\cos y - e^{-y}\cos x \qquad h(x, y) = \ln\sqrt{x^2 + y^2}$$

Question 3:

(a) Newton's law of universal gravitation states that the magnitude of the gravitational force F between two objects is given by

$$F = G\frac{m_1 m_2}{r^2},$$

where G is the gravitational constant, m_1 and m_2 are the masses of the two objects, and r is the distance between the objects. Here $G \approx 6.674 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$. A team of amateur astronomers have estimated that $m_1 = 2 \times 10^{24}$ kg, $m_2 = 5 \times 10^{23}$ kg, and $r = 10^{10}$ m, with a maximum relative error[1] of 3% in each measurement. Use differentials to estimate the maximum relative error in the calculated force F.

(b) Use differentials to approximate the number $(1.98)^3\left((3.03)^2 - \frac{1}{(1.01)^3}\right)$. It may help to consider a suitable function $f(x, y, z)$ at a suitable point $P(a, b, c)$.

[1] The *relative error* of a value x is defined as the ratio of the absolute error to the true value of the quantity. If the true value is x and the measured value is x_0, then the percent relative error would be $100 \cdot |x - x_0| / |x|$.

Question 4: Suppose $f(x, y)$ is a twice continuously differentiable function with function values measured in the table below:

$y \backslash x$	-1	0	1	2	3
-1	11	12	15	14	13
0	13	16	17	18	20
1	20	22	22	19	18
2	27	26	25	22	20
3	32	28	28	27	26

(a) Approximate f_x and f_y at the point $(1, 2)$.

(b) Approximate f_{xy} and f_{xx} at the point $(1, 2)$.

(c) Using the table directly, approximate the directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = \langle 1, -1 \rangle$.

(d) Using the gradient vector, approximate the directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = \langle 1, -1 \rangle$. Does your answer agree with the previous part? Explain.

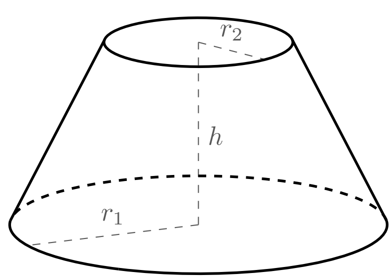
Question 5: Consider the ellipsoid $x^2 + 2y^2 + 4z^2 + xy + 4yz = 71$.

(a) Show that the points on the ellipsoid where the tangent plane is vertical (parallel to the z-axis) constitute the intersection of the ellipsoid with a certain plane, and find the equation of that plane.

(b) Consider the point $P(1, 2, 3)$ (check that it is on the ellipsoid!). Since this point is not among those of part (a), a piece of the ellipsoid containing P is the graph of a function $g(x, y)$. Use implicit differentiation to compute g_x and g_y in terms of $(x, y, g(x, y))$, as well as $g_x(1, 2)$ and $g_y(1, 2)$.

Question 6:

(a) A truncated right circular cone has a height, and two radii (see picture below). The smaller radius of this cone is decreasing at a constant rate of 1 cm/s, the larger radius is increasing at a constant rate of 2 cm/s, and the height of the cone is decreasing at a constant rate of 3 cm/s. At what rate is the volume of the cone changing when the smaller radius is 10 cm, the larger radius is 15 cm, and the height is 8 cm?



(b) If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

Extra Credit: In this question let's explore how much more interesting the notion of continuity and differentiability can be in higher dimensions. First, let's look at continuity:

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ (this notation means all points in the plane except for the origin):

$$f(x,y) = \frac{xy}{x^2+y^2}$$

By letting $x = r\cos\theta$ and $y = r\sin\theta$, describe the level sets of f. Explain why there is no value we can assign to $f(0,0)$ that would make this function continuous.

(b) Now consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined everywhere on $\mathbb{R}^2$ by

$$g(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Using the same basic trick as in part (a), explain how you know

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

(c) One last interesting example. Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$h(x,y) = \frac{x^2 y}{x^4+y^2}$$

Is there a value you can assign to $h(0,0)$ to make h continuous at the origin? Justify your work.

Now let's turn to differentiability.

(d) Now consider the function $p(x,y) = (xy)^{1/3}$. Compute $p_x(x,0)$ for any x and $p_y(0,y)$ for any y. In particular, compute both p_x and p_y at $(0,0)$.

(e) Along the positive x-axis, does this function have a tangent plane? What is it? What about along the positive y-axis?

(f) Does this function have a tangent plane at the origin? Explain.