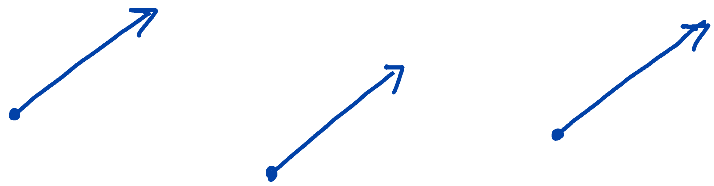
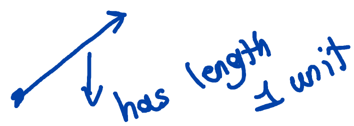
2 Section 12.2: Vectors

To represent objects in 2D, 3D, and even higher dimension spaces, we introduce **vectors**.

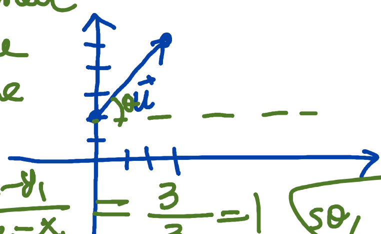
- **Definition:** A vector $\vec{v}$ is an object with a **magnitude** and a **direction**.



- **Unit vector:** A vector $\vec{v}$ is said to be a unit vector if its **magnitude is 1**



Example: Sketch a vector $\vec{u}$ in $\mathbb{R}^2$ with start point $(0, 2)$ and end point $(3, 5)$. Find the **magnitude** and describe the **direction** of the vector $\vec{u}$.



The angle formed by $\vec{u}$ and the horizontal line satisfies

$\tan(\theta) = \frac{y_2-y_1}{x_2-x_1} = \frac{3}{3} = 1$ so, $\theta = \frac{\pi}{4}$ or 45°

$\|\vec{u}\|$ = Distance between $(0,2)$ and $(3,5)$

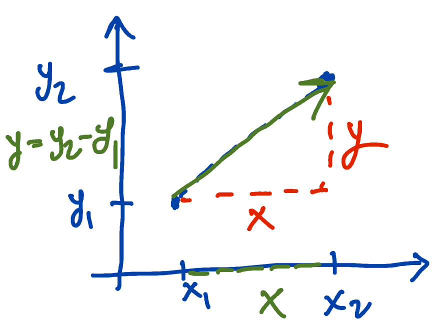
$= \sqrt{(3-0)^2+(5-2)^2} = \sqrt{9+9} = 3\sqrt{2}$

- **Component form of a vector** Every 2-D (resp. 3-D) vector $\vec{v}$ can be represented as a point in the coordinate system:

$$\vec{v} = < x, y > \text{ (resp. } \vec{v} = < x, y, z >).$$

Such representation is called the component form of the vector.

How to find the component for of a vector using its start and end point?



$x = x_2 - x_1$

I can define $\vec{v}$ by

$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ where $x = x_2 - x_1$ and $y = y_2 - y_1$

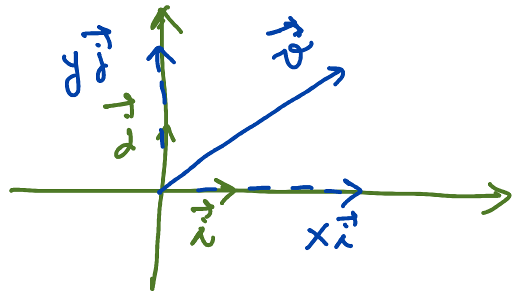
$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$, and $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$

We can also write

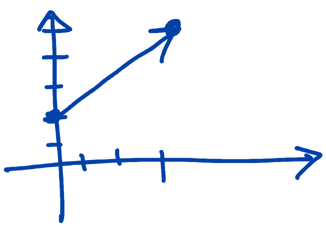
$$\vec{v} = x\vec{i} + y\vec{j} \text{ (resp. } \vec{v} = x\vec{i} + y\vec{j} + z\vec{k}),$$

where

$\vec{i} =< 1, 0 >$ and $\vec{j} =< 0, 1 >$ (resp. $\vec{i} =< 1, 0, 0 >, \vec{j} =< 0, 1, 0 >$, and $\vec{k} =< 0, 0, 1 >$)



Example: Consider again the vector $\vec{u}$ with start point $(0, 2)$ and end point $(3, 5)$. Find its component form.



Then $\vec{u} = \begin{pmatrix} 3-0 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

or $\vec{u} = 3\vec{i} + 3\vec{j}$.

- In general, let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. We denote by $\overrightarrow{PQ}$ the vector with start point P and end point Q. Find the component form of $\overrightarrow{PQ}$.

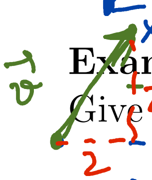
$$\overrightarrow{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

- **Equivalent vectors:** Two vectors $\vec{u}$ and $\vec{v}$ are **equivalent** if their component forms are identical. This is equivalent to say that the vector have the same magnitude and direction.

$\vec{u} = \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

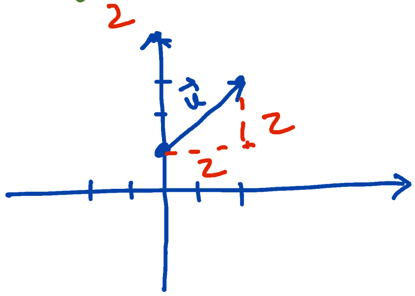
+2

+2



Example: Consider the vector $\vec{u}$ with start point $(0, 1)$ and end point $(2, 3)$. Give a vector $\vec{v}$ with start point $(-2, 5)$ and equivalent to $\vec{u}$.

+2 +2

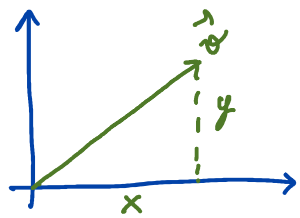


$\vec{v}$ must end at $(0, 7)$

Given the component form of a vector, how to find its magnitude?

Let $\vec{v} = \langle x, y, z \rangle = x\vec{i} + y\vec{j} + z\vec{k}$. so, $\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$

(2-D picture)



- **Example:** Consider the vector $v = 2\vec{i} - \vec{j} - 2\vec{k}$.
  - Find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$. where $\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

so, $\boxed{\vec{u} = \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}}$

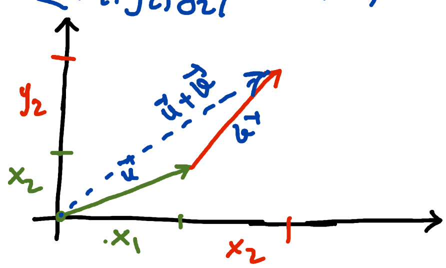
  - Find the vector $\vec{w}$ of length 7 in the **opposite** direction of $\vec{v}$.

$\vec{w} = (-7)\vec{u}$. $\quad \vec{w} = -\frac{14}{3}\vec{i} + \frac{7}{3}\vec{j} - \frac{14}{3}\vec{k}$

- **Adding two vectors**

Let $\vec{u} = \langle x_1, y_1, z_1 \rangle$, and $\vec{v} = \langle x_2, y_2, z_2 \rangle$. Then, $\vec{u} + \vec{v} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$.

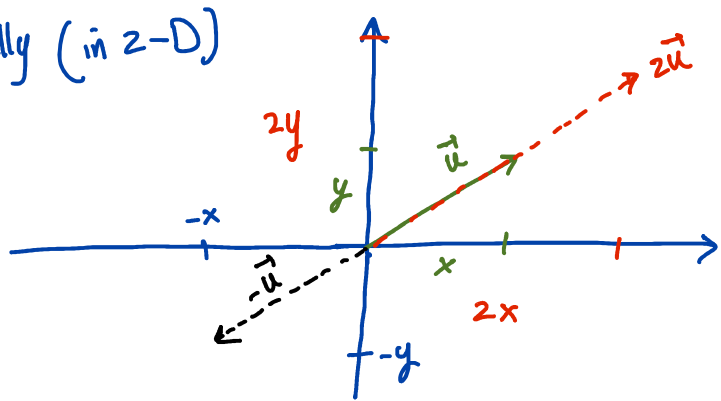
Graphically (in 2-D):



- **Multiplying a vector by a scalar**

Let $\vec{u} = \langle x, y, z \rangle$. Then $a\vec{u} = \langle ax, ay, az \rangle$

Graphically (in 2-D)



- **Linear combination of vectors**

If $\vec{u} = \langle x_1, y_1, z_1 \rangle$, $\vec{v} = \langle x_2, y_2, z_2 \rangle$, and a and b are scalars,

then $a\vec{u} + b\vec{v} = \langle ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2 \rangle$

- **Properties of vectors addition and multiplying by a scalar** (here a, b are scalars, and $\vec{u}, \vec{v}, \vec{w}$ are vectors)

(Try examples at home).

  - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
  - $\vec{u} + \vec{0} = \vec{u}$
  - $\vec{u} + \vec{-u} = \vec{0}$
  - $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
  - $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
  - $(ab)\vec{u} = a(b\vec{u})$
  - $1\vec{u} = \vec{u}$