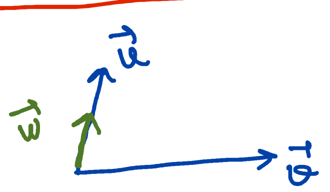
3 Section 12.3: The Dot Product

- The **dot product** (or scalar product) of two vectors $\vec{u} = (a_1, b_1, c_1)$ and $\vec{v} = (a_2, b_2, c_2)$ is defined by

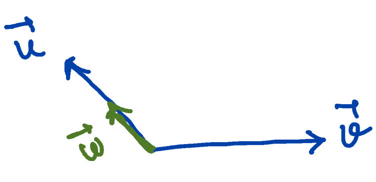
$$\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

- **Properties:** (Try some examples at home)
  - $\vec{u} \cdot \vec{u} = a_1^2 + b_1^2 + c_1^2 = \|\vec{u}\|^2$
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
  - $(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (a\vec{v})$
  - $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$
  - $(\vec{u} + \vec{v}) \cdot \vec{w} = (\vec{u} \cdot \vec{w}) + (\vec{v} \cdot \vec{w})$
  - $\vec{u} \cdot (\vec{v} \cdot \vec{w}) =$ undefined BUT $(\vec{v} \cdot \vec{w})\, \vec{u}$ well-defined.

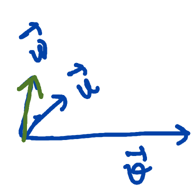
Some examples:



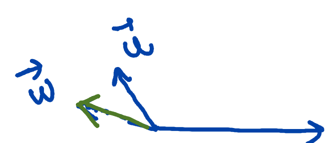
$0 < \vec{w} \cdot \vec{v} < \vec{u} \cdot \vec{v}$



$\vec{u} \cdot \vec{v} < \vec{w} \cdot \vec{v} < 0$



$\vec{u} \cdot \vec{v} > \vec{w} \cdot \vec{v}$



$\vec{u} \cdot \vec{v} > \vec{w} \cdot \vec{v}$

What does the dot product say about the relationship between two given vectors?

- **Theorem:** Let θ be the angle between the two vectors $\vec{u}$ and $\vec{v}$ ($0 \leq \theta \leq \pi$). Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$$

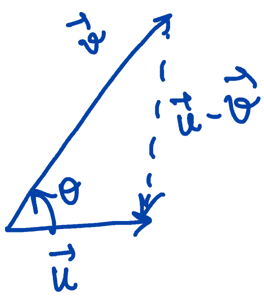
and

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$$

How can we prove this?

Hint: use the Law of Cosine:

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\|\vec{u}\| \|\vec{v}\| \cos(\theta)$$



Now $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u} \cdot \vec{v}$.

Then, $\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u} \cdot \vec{v}$

Thus, $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$

Interpretation:

(See examples earlier

- **Definition of orthogonal vectors:** Two vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal if $\vec{u} \cdot \vec{v} = 0$.

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta)$

If $\vec{u}$ and $\vec{v}$ are not zero vectors, then $\vec{u} \cdot \vec{v} = 0$ if $\cos(\theta) = 0$

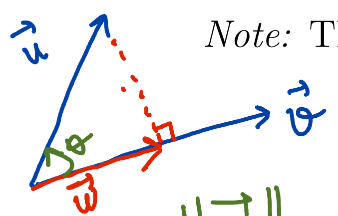
Then $\theta = \frac{\pi}{2}$ (or 90°).

What else does the dot product say about the relationship between two given vectors?

- **Proposition:** Let $Proj_{\vec{v}}\vec{u}$ be the vector projection of the vector $\vec{u}$ onto the vector $\vec{v}$. Then,

$$\vec{w} = Proj_{\vec{v}}\vec{u} = \boxed{\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}}\vec{v}$$

Note: The scalar $Comp_{\vec{v}}\vec{u} := \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$ is called the scalar projection of $\vec{u}$ onto $\vec{v}$.



$$Proj_{\vec{v}}\vec{u} = \left(\ ?? \ \right) \frac{\vec{v}}{\|\vec{v}\|}$$

$$\cos(\theta) = \frac{\|\vec{w}\|}{\|\vec{u}\|}$$

so, $\|\vec{w}\| = \|\vec{u}\| \cdot \cos(\theta)$

$= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta) \frac{1}{\|\vec{v}\|}$

$= \vec{u} \cdot \vec{v} \cdot \frac{1}{\|\vec{v}\|}$.

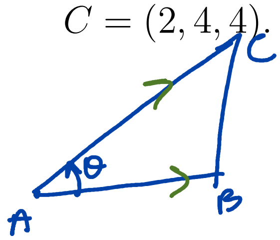
so $Proj_{\vec{v}} u = \vec{u} \cdot \vec{v} \frac{1}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|}$

$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$

Interpretations:

$\|Proj_{\vec{v}}\vec{w}\| > \|Proj_{\vec{v}}\vec{u}\|$ $\quad$ $\|Proj_{\vec{v}}\vec{w}\| > \|Proj_{\vec{v}}\vec{u}\|$ $\quad$ $\|Proj_{\vec{v}}\vec{w}\| < \|Proj_{\vec{v}}\vec{u}\|$ $\quad$ $\|Proj_{\vec{v}}\vec{w}\| > \|Proj_{\vec{v}}\vec{u}\|$

- **Example:** Consider the triangle with vertices $A = (2, 0, 1)$, $B = (4, 2, 0)$, $C = (2, 4, 4)$. Find the angle at A.



$\vec{AC} \cdot \vec{AB} = \|\vec{AC}\| \cdot \|\vec{AB}\| \cos(\theta)$

$\vec{AC} = \langle 0, 4, 3 \rangle$ and $\vec{AB} = \langle 2, 2, -1 \rangle$