4 Section 12.4: The Cross Product

- **Definition:** Let $\vec{u} = (a, b, c)$ and $\vec{v} = (d, e, f)$ be two vectors. The **cross product** $\vec{u} \times \vec{v}$ is a **vector** defined by

$$\vec{u} \times \vec{v} = \langle bf - ce, -(af - cd), ae - bd \rangle$$

.

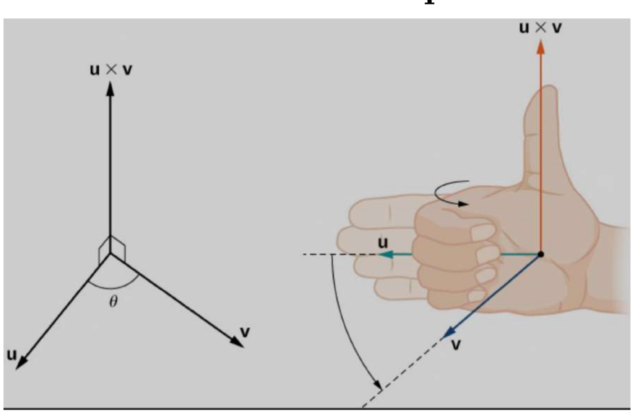
  - In matrix determinant notation:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \vec{i} \begin{vmatrix} b & c \\ e & f \end{vmatrix} - \vec{j} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \vec{k} \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= \vec{i}\,(bf - ec) - \vec{j}\,(af - dc) + \vec{k}\,(ae - bd)$$

  - **Note:** *There is no cross product in 2-D. But if you wish to find the cross product of two vectors in $\mathbb{R}^2$, then you can assume that the z component is zero.*

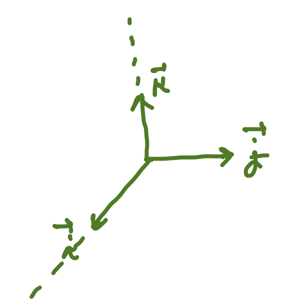
- **Direction of the cross product vector: Right Hand Rule**

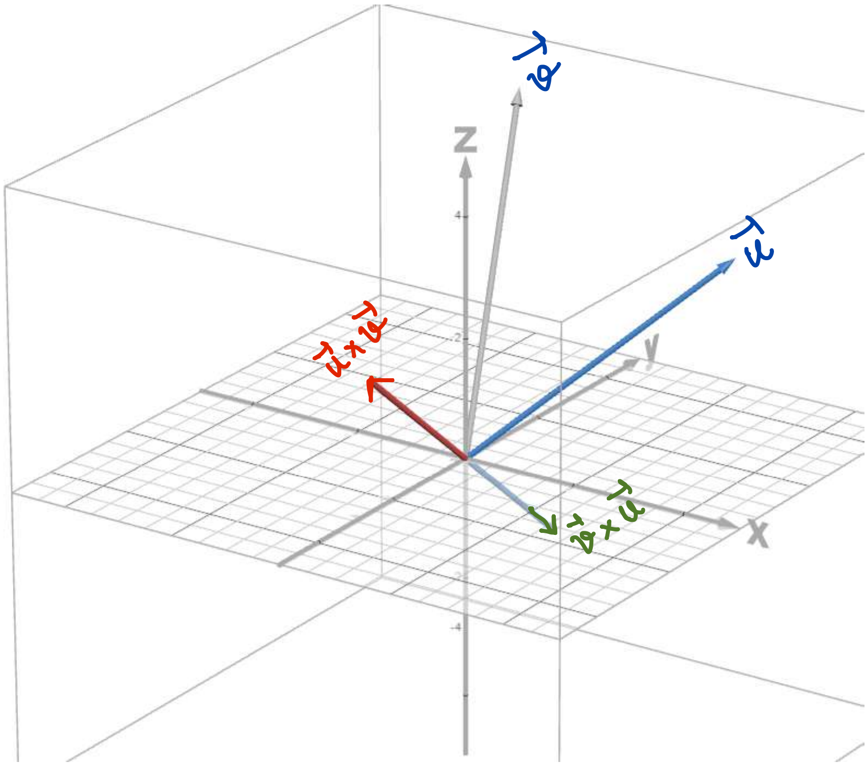


Examples:

$$\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$$

$$\vec{j} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}$$





- **Algebraic properties of the cross product**
  - $\vec{u} \times \vec{u} = \vec{0}$
  - $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
  - $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
  - $(r\vec{u}) \times \vec{v} = r(\vec{u} \times \vec{v})$
  - $\vec{u} \times \vec{0} = \vec{0}$

Try some examples to verify these properties.

What does the cross product say about the relationship between two given vectors?

- **Theorem: The cross product $\vec{u} \times \vec{v}$ is orthogonal to both of the vectors $\vec{u}$ and $\vec{v}$**

  Why? *try finding a dot product...*

By direct computation, you should be able to show that

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$

- **Theorem:** Given two vectors $\vec{u}$ and $\vec{v}$, Let θ be the angle between them $(0 \leq \theta \leq \pi)$. Then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin(\theta)$$

  and

  How can we prove this?

  Hint: by direct computation, you can find: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$, *then continue from there...*

Then, $\|\vec{u} \times \vec{v}\|^2 = \|u\|^2 \cdot \|\vec{v}\|^2 - \|u\|^2 \cdot \|\vec{v}\|^2 \cdot \cos^2(\theta)$

$= \|\vec{u}\|^2 \cdot \|\vec{v}\| (1 - \cos^2(\theta))$

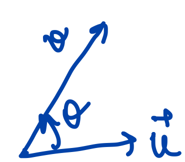
$= \|\vec{u}\|^2 \cdot \|\vec{v}\|^2 \sin^2(\theta)$

- **Corollary:** Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = 0$

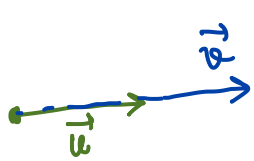
$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \sin(\theta)$

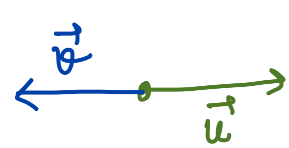
If $\|\vec{u}\|$ and $\|\vec{v}\|$ are not zero, then

$\|\vec{u} \times \vec{v}\| = 0$ is equivalent to $\sin(\theta) = 0$



Then $\theta = 0$ or $\theta = \pi$





Interpretation:

- **Example:** Find two unit vectors which are orthogonal to $\vec{u} = (-1, 0, 3)$.

Let $\vec{a} = (3, 1, 1)$ Then $\vec{u} \cdot \vec{a} = (-1)(3) + (1)(0) + (1)(3) = 0$

$\|\vec{a}\| = \sqrt{9+1+1} = \sqrt{11}$. Then $\vec{\vartheta} = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

we found the first vector.

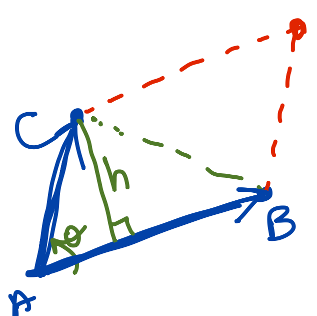
Now, let $\vec{b} = \vec{u} \times \vec{\vartheta}$ then divide by its magnitude to get the second vector.

or $\vec{\vartheta} \times \vec{u}$.

- **Finding the area of a parallelogram using cross product.**

Example: Let $A = (1, 2, 7)$, $B = (-1, 1, 5)$, $C = (3, 1, 1)$

Complete the parallelogram and find its area.



solution:

$$\text{Area} = 2\left(\frac{\|\vec{AB}\| \cdot h}{2}\right) = \|\vec{AB}\| \cdot h$$

But $\sin(\theta) = \frac{h}{\|\vec{AC}\|}$. so, $h = \|\vec{AC}\| \sin(\theta)$.

$$\text{so, } \boxed{\text{Area} = \|\vec{AB}\| \cdot \|\vec{AC}\| \sin(\theta) = \|\vec{AB} \times \vec{AC}\|}$$

so, $\vec{AB} = \langle -2, -1, -2 \rangle$, $\vec{AC} = \langle 2, -1, -6 \rangle$ Then $\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -2 \\ 2 & -1 & -6 \end{vmatrix}$

$\vec{AB} \times \vec{AC} = \vec{i}\,(6-2) - \vec{j}\,(12+4) + \vec{k}\,(2+2) = 4\vec{i} - 16\vec{j} + 4\vec{k}$

and $\|\vec{AB} \times \vec{AC}\| = \sqrt{4^2 + 16^2 + 4^2}$.