6 Section 12.6: Cylinders and Quadric Surfaces

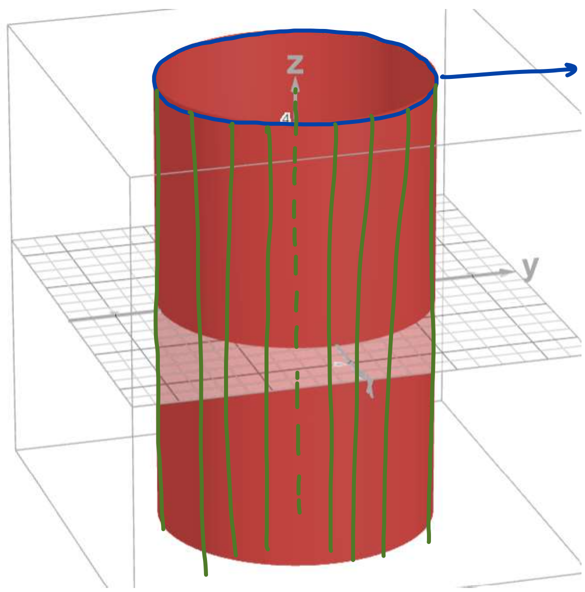
- **What is a cylinder?**

- A **Cylinder** is a surface that consists of all lines (called **rullings**) that are parallel to a given line and pass through a given plane curve.

  - **Circular Cylinders**

    $x^2 + y^2 = 9$

    Take the circle in the xy-plane centered at $(0, 0, 0)$ *and of radius* 3 *and move it parallel to the z-axis.*

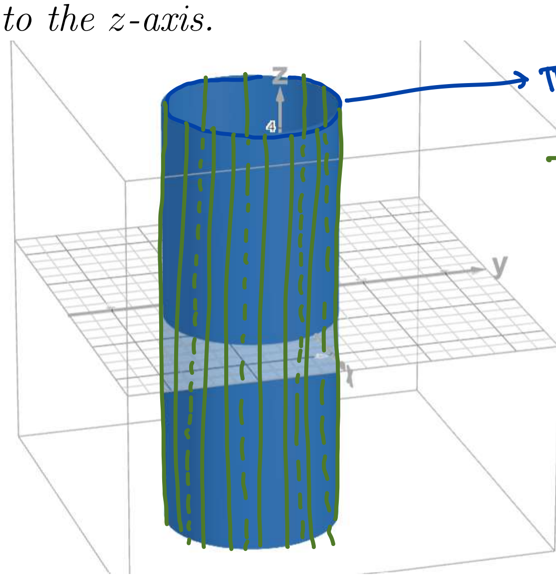


The plane curve in this example is a circle

The lines are parallel to the z-axis.

$(x-1)^2 + (y-1)^2 = 4$

Take the circle in the xy-plane centered at $(1, 1, 0)$ *and of radius* 2 *and move it parallel to the z-axis.*



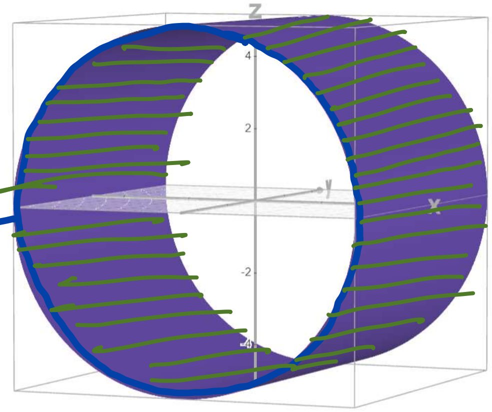
The plane curve.

The lines are parallel to the z-axis.

$x^2 + z^2 = 25$

Take the circle in the xz-plane centered at $(0,0,0)$ and of radius 5 and move it parallel to the y-axis.

The plane curve is a circle



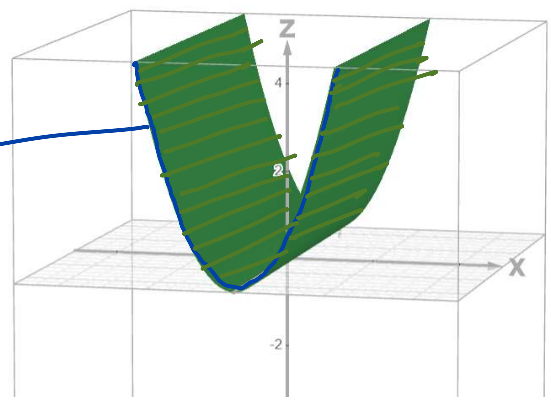
The lines are parallel to the y-axis.

- **Parabolic Cylinders**

$z = x^2$

Take the parabola $z = x^2$ in the xz-plane and move it parallel to the y-axis.

The plane curve is a parabola

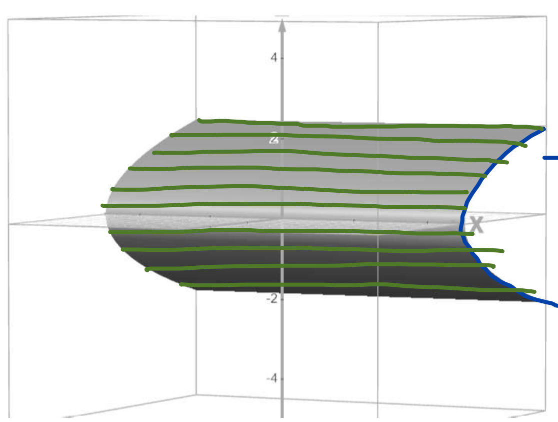


The lines are parallel to the y-axis.

$y = z^2$

Take the parabola $y = z^2$ in the yz-plane and move it parallel to the x-axis.

The lines are parallel to the X-axis.



the space curve is a parabola

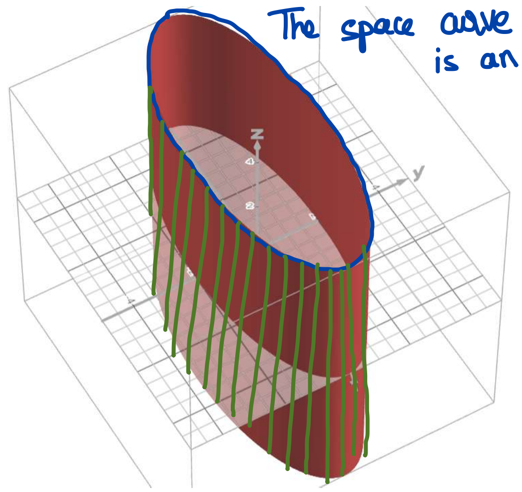
- **Elliptic cylinders:**

  $\frac{x^2}{25} + \frac{y^2}{4} = 1$

  Take the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ *in the* xy*-plane and move it parallel to the* z*-axis.*

The space above is an ellipse

The lines are parallel to the z-axis.



- **Traces**

  A **trace** of a surface is the curve that lies in a planar cross-section.

- **What are quadric surfaces?** A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The **standard form** for the equation of a quadratic surface (after translations and rotations) is

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0,$$

  where A, B, C, I, and J are constants.

- **Ellipsoid**

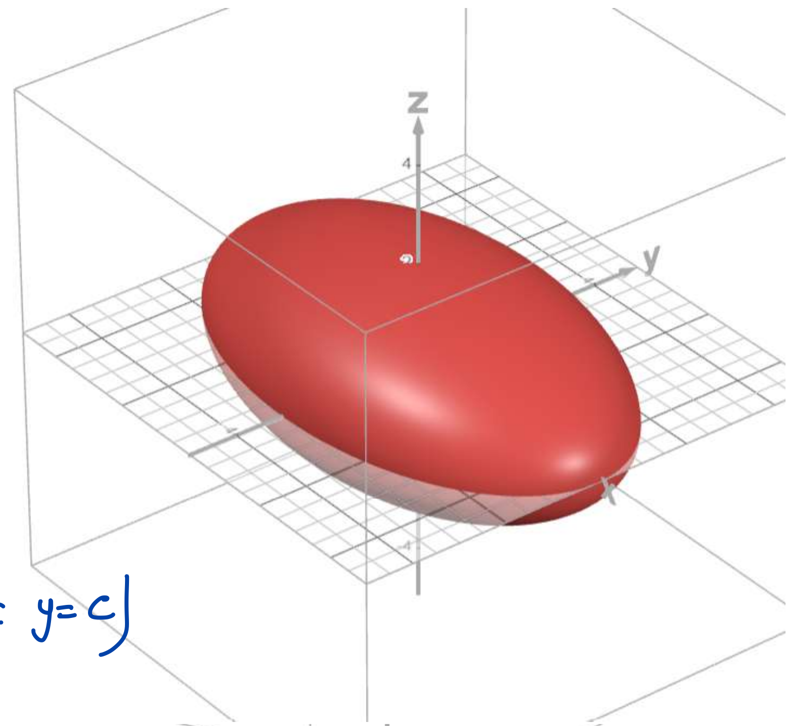
$$\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Conditions:

$1 - \frac{x^2}{25} \geqslant 0$; $-5 \leqslant x \leqslant 5$

$1 - \frac{y^2}{9} \geqslant 0$; $-3 \leqslant y \leqslant 3$

$1 - \frac{z^2}{4} \geqslant 0$; $-2 \leqslant z \leqslant 2$



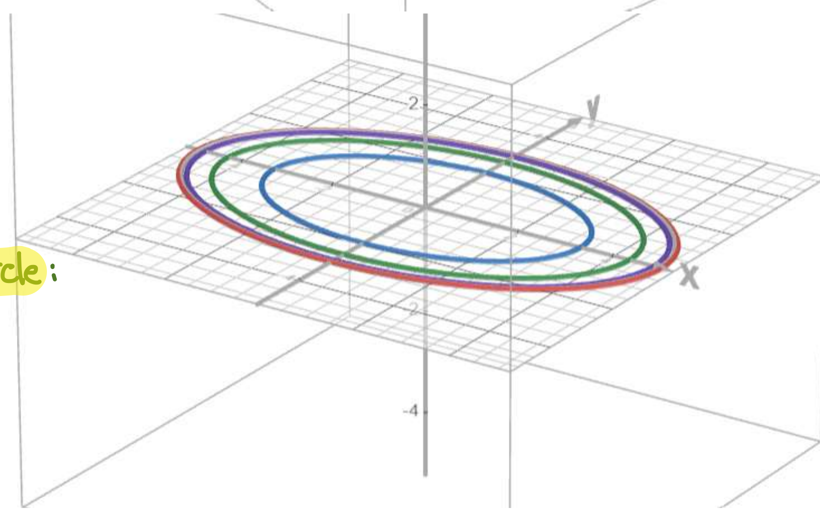
Horizontal traces: $(z = c)$

$\frac{x^2}{25} + \frac{y^2}{9} = 1 - \frac{c^2}{4}$

Ellipses.

Vertical traces: $(x = c$ or $y = c)$

are also ellipses.



NOTE:

When $a = b = c$,

the surface is a circle:

$x^2 + y^2 + z^2 = a^2$

■ **Elliptic Paraboloid**

$$\frac{z}{2} = \frac{x^2}{25} + \frac{y^2}{9}$$

Conditions: $z \geq 0$.

Horizontal traces $(z = c)$

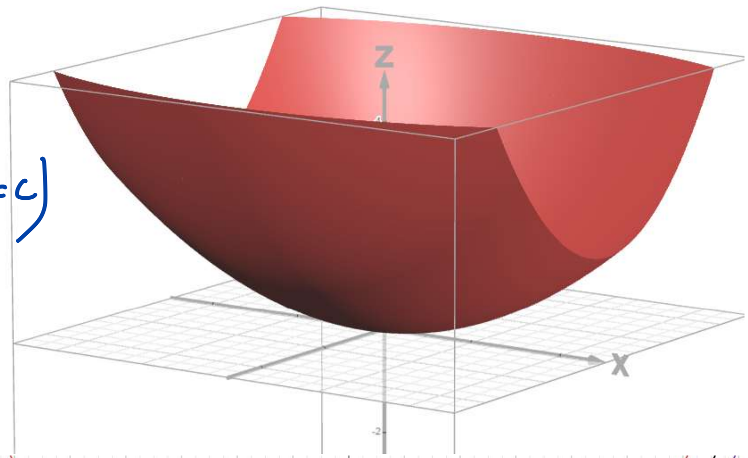
are ellipses.

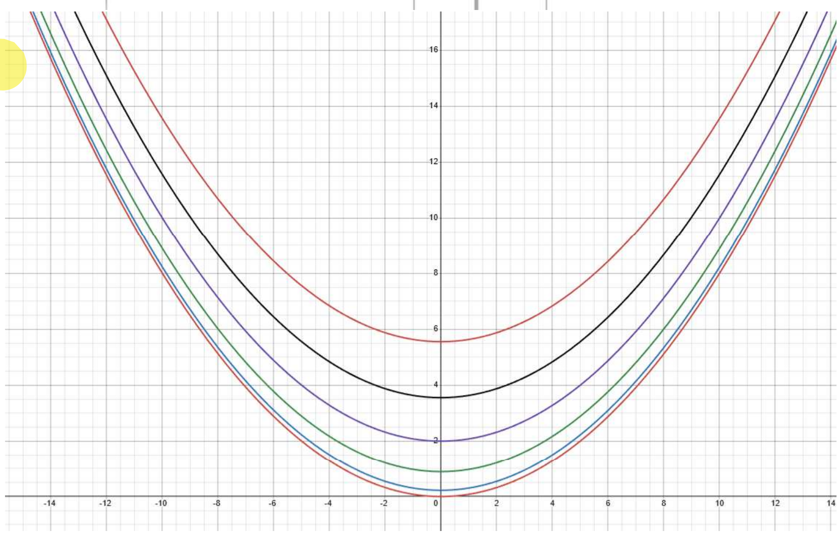
Vertical traces $(x = c$ or $y = c)$

For $x = c$, $\frac{z}{2} = \frac{c^2}{25} + \frac{y^2}{9}$

$$z = \frac{2}{9} y^2 + \frac{2c^2}{25}$$

This is a parabola





Vertical traces

Hyperbolic Paraboloid

$$\frac{z}{2}=\frac{x^2}{25}-\frac{y^2}{9}$$

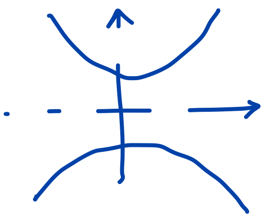
Horizontal traces: $(z=c)$

$$\frac{x^2}{25}-\frac{y^2}{9}=\frac{c}{2}$$

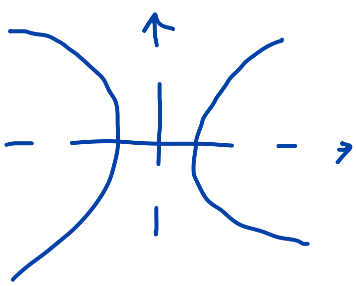
This is a hyperbola

$$\frac{y^2}{9}=\frac{x^2}{25}-\frac{c}{2}$$

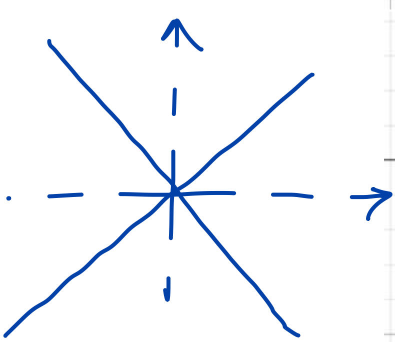
when $c<0$ the trace looks like:

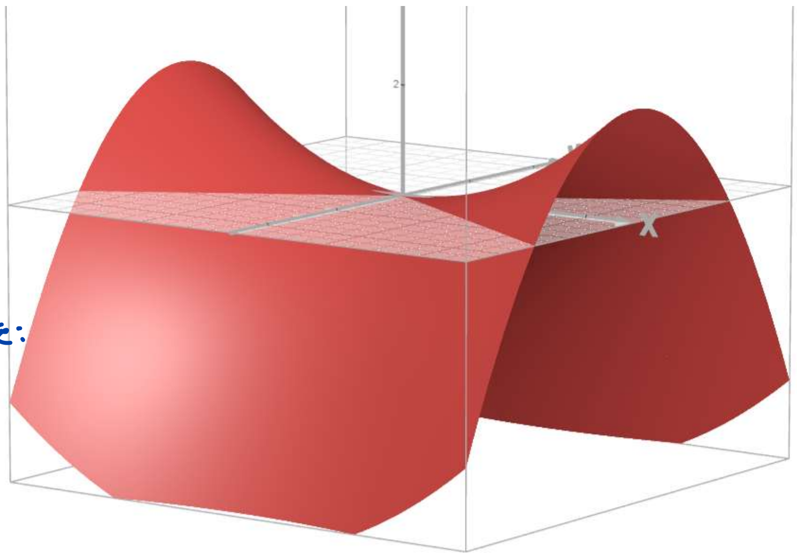


when $c>0$, the trace looks like:



when $c=0$, the trace looks like:



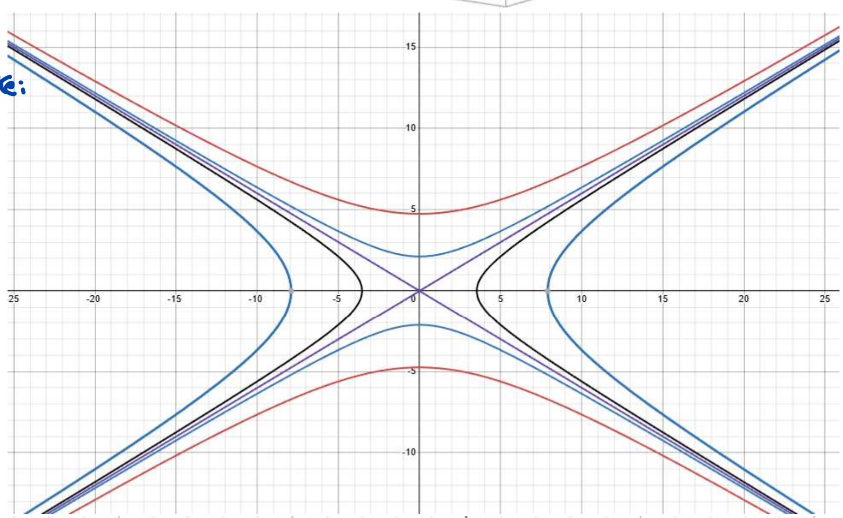


Vertical traces: $(x=c \text{ or } y=c)$

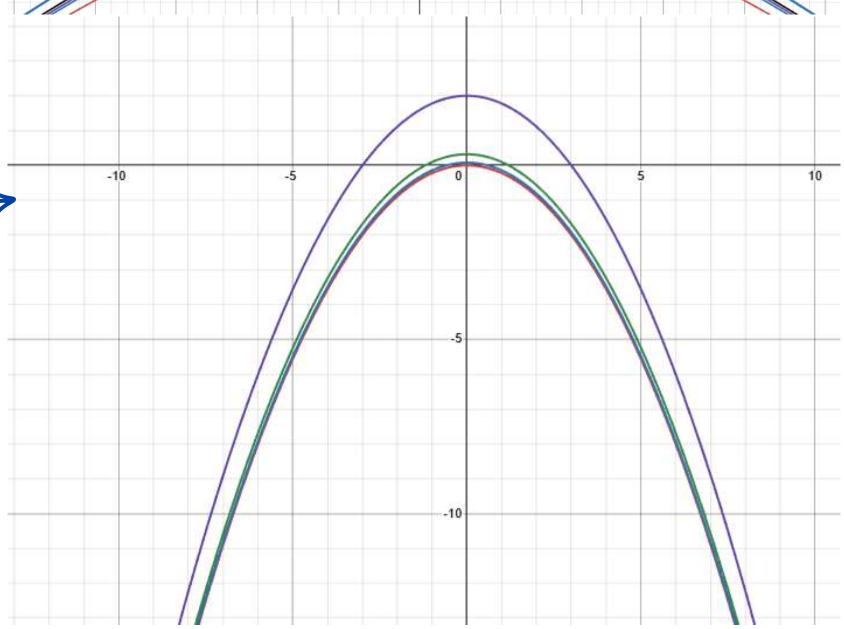
For $x=c$,

$$z=-\frac{2}{9}y^2+\frac{2}{25}c^2$$

This is a Parabola



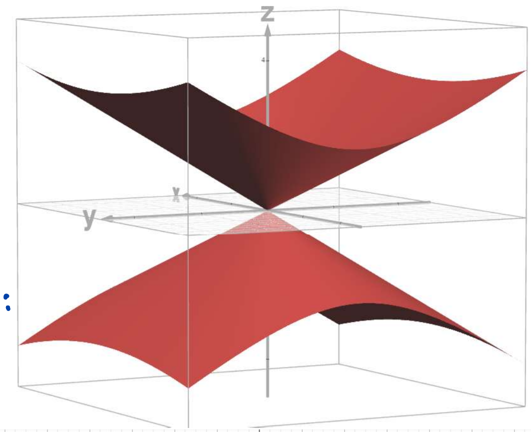
Horizontal traces



Vertical traces

- **Cone**

$$\frac{z^2}{4} = \frac{x^2}{25} + \frac{y^2}{9}$$



Horizontal traces ($z=C$):

$$\frac{x^2}{25} + \frac{y^2}{9} = \frac{c^2}{4}.$$

This is an ellipse.

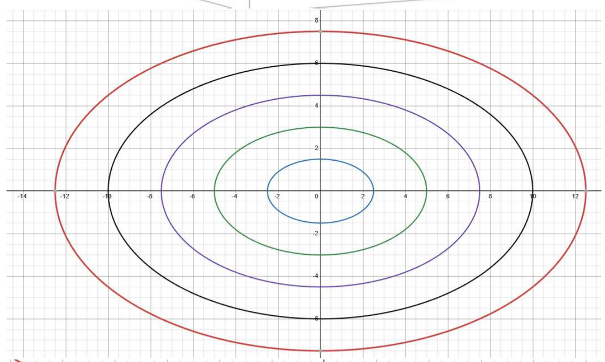
Vertical traces ($x=c$ or $y=c$):

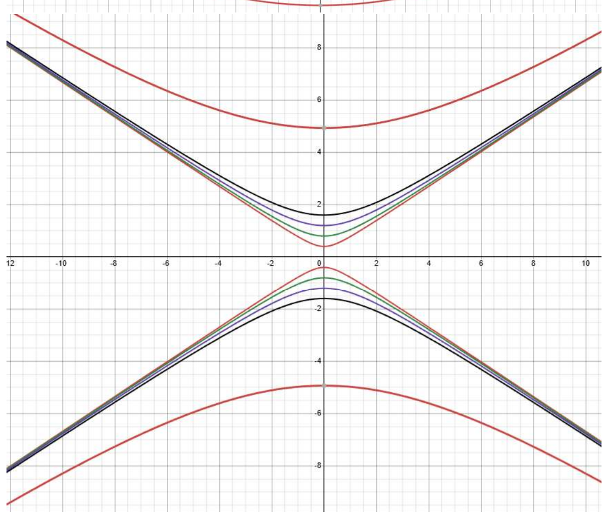
For $x=c$,

$$z^2 = \frac{4}{25}c^2 + \frac{4}{9}y^2$$

If $c=0$, pair of 2-lines

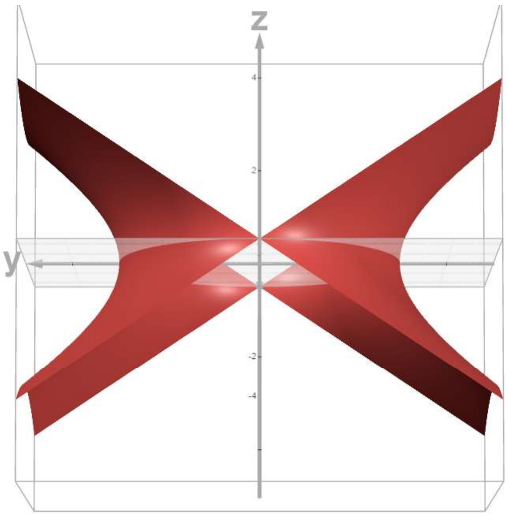
If $c\neq 0$, hyperbola



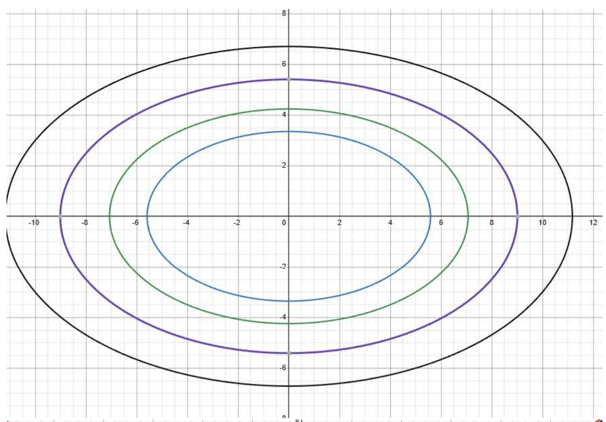


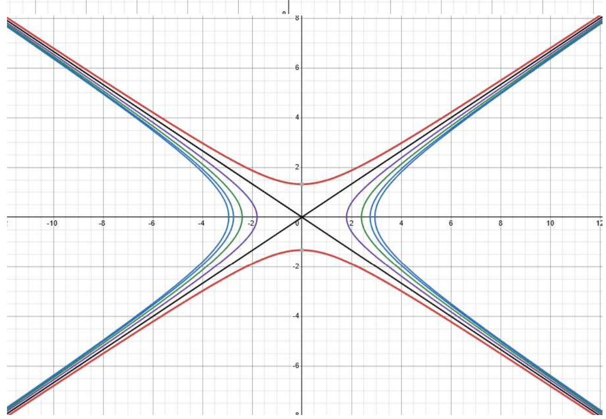
- **Hyperboloid of One Sheet**

$$\frac{x^2}{25} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$



(Do the work at home)

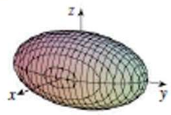
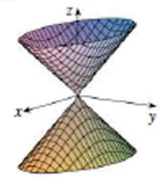
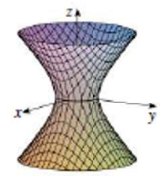
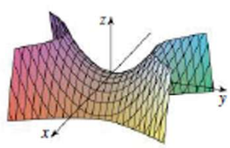
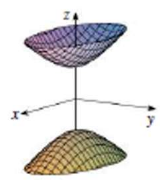




▪ Hyperboloid of Two Sheet

Do the work, sketch the surface and the traces at home.

$$\frac{x^2}{25} + \frac{y^2}{9} - \frac{z^2}{4} = -1$$

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.