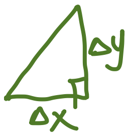
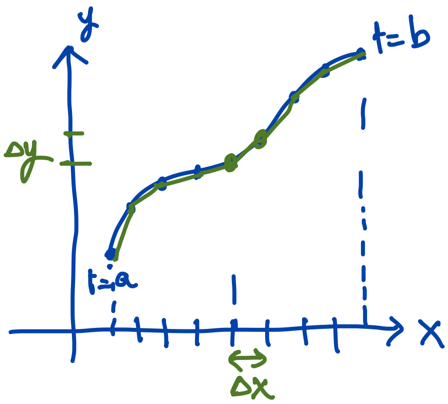
2 Section 13.3: Arclength and Curvature



- In 2-D, when we considered a curve parametrized by $< \overset{x(t)}{f(t)}, \overset{y(t)}{g(t)} >$ for $a \leq t \leq b$, and we wanted to find the length of the curve for t in this range, we derived the following formula for arc length:



$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2} \cdot (\Delta t)$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The length of a curve in the space is defined exactly the same way.

- **Prop:** Given a space curve parametrized by $\vec{r}(t) =< f(t), g(t), h(t) >$ for $a \leq t \leq b$, the arclength between $\vec{r}(a)$ and $\vec{r}(b)$ is equal to

$$\int_a^b \|\vec{r'}(t)\| dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

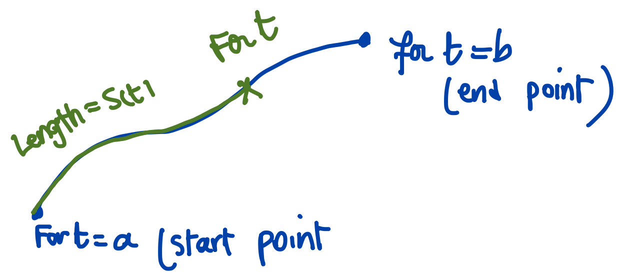
(Similar idea of proof works here).

- **Definitions:** Consider a curve C parametrized by a vector function $\vec{r}(t)$ define for $t \geq a$.

  - The **arc length function** at t is

$$s(t) := \int_a^t |\vec{r'}(u)| du$$

    We see that $s(t)$ represents the length of the part of the curve C between $r(a)$ and $r(t)$.



    Note: By the Fundamental Theorem of Calculus,

$$\frac{dS}{dt} = \|\vec{r'}(t)\|$$

  - An **arc length parametrization** of $\vec{r}(t)$ is a re-parametrization of $\vec{r}(t)$ in terms of s.

    Doing that, the arc length of $\vec{r}(s)$ from the starting point to s is always exactly s!

  - Its **curvature** at t is

$$k(t) := \|\frac{d\vec{T}}{ds}\|,$$

    where $\vec{T}$ is the unit tangent vector, and s is the arc length.

    The curvature of a curve C at a given point measures how quickly the curve changes direction at that point. In other words, it indicates how tightly the curve bends.

    For examples, lines have curvature 0, and circles of radius R have curvature $\frac{1}{R}$.

Usually, it is hard to do the computations using this definition. Instead, notice that

$$k(t) := \|\frac{d\vec{T}}{dt}\frac{dt}{ds}\| = \frac{\|\vec{T}'(t)\|}{\|\vec{r'}(t)\|}$$

which can be shown to be equal to

$$k(t) = \frac{\|\vec{r'}(t) \times \vec{r''}(t)\|}{\|\vec{r'}(t)\|^3}$$

- Its **Torsion** indicates how much a curve twists out of a plane.

- **Example:**

  Reparametrize the helix $r(x) = \cos(t)i + \sin(t)j + tK$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.