3 Section 13.4: Motion in space

- **Definitions:**

  Let $\vec{r}(t) =< f(t), g(t), h(t) >$ describe the **position** of an object at time t. then

  - The **velocity** of the object is $v(t) = \vec{r'}(t) =< f'(t), g'(t), h'(t) >$.
  - The **speed** of the object is $\|v(t)\| = \|\vec{r'}(t)\|$.
  - The **acceleration** of the object is $v'(t) = \vec{r''}(t)$.

- **Example 1:** Suppose that two objects A and B are moving along the unit circle. The position of object A is

$$\vec{r_A}(t) =< \cos(t), \sin(t), 0 >$$

note: $(\cos(t))^2 + (\sin(t))^2 = 1 \Rightarrow x^2+y^2=1$

and the position of object B is

$$\vec{r_B}(t) =< \cos(t/2), \sin(t/2), 0 > .$$

(same here)

The parameter t represents the time. Which object is going faster on the unit circle? A

The trajectories of A and B are the unit circle.

$\vec{v}_A(t) = \langle -\sin(t), \cos(t), 0 \rangle$, $\vec{v}_B(t) = \langle -\frac{1}{2}\sin\left(\frac{t}{2}\right), \frac{1}{2}\cos\left(\frac{t}{2}\right), 0 \rangle$

speed of A $= \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0}$ $= 1$

speed of B $= \sqrt{\frac{1}{4}\sin^2\left(\frac{t}{2}\right) + \frac{1}{4}\cos^2\left(\frac{t}{2}\right) + 0}$ $= \sqrt{\frac{1}{4}} = \frac{1}{2}$

we see that speed A > speed B

- **Example 2:** Suppose that the positions of two objects A and B can be described by

$$\vec{v}_A(t) = \langle 1, 1, -1 \rangle \Leftarrow \vec{r_A}(t) =< t+1, t-3, -t >$$

start point: $(1, -3, 0)$

Direction vector: $\langle 1, 1, -1 \rangle$

and B is

$$\vec{v}_B(t) = \langle -1, -1, 1 \rangle \Leftarrow \vec{r_B}(t) =< -s+1, -s-3, s > .$$

start point $(1, -3, 0)$

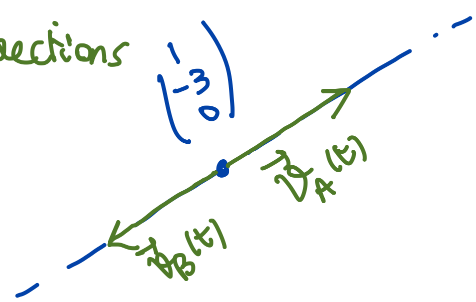
Direction vector: $\langle -1, -1, 1 \rangle$

Here, t represents the time. Compare the positions, velocities, speeds, and accelerations of the two objects at a given time.

The velocities has same magnitude but opposite directions

so, A and B have the same speed.

The accelerations of A and B are zero.



$\vec{r}(0)$

- **Example 3:** A particle was fired at $(0, 0, 6)$ with initial velocity $\vec{v_0} = (3, 4, 1)$, and moves with constant acceleration $\vec{a} = (0, 0, -2)$. Find the position of the particle where it hits the ground ($z = 0$).

FTC

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)\, du = \langle 3,4,1\rangle + \int_0^t \langle 0,0,-2\rangle\, du$$

$$\vec{v}(t) = \langle 3,4,1\rangle + \langle 0,0,-2t\rangle = \langle 3,4,1-2t\rangle$$

FTC.

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)\, du = \langle 0,0,6\rangle + \int_0^t \langle 3,4,1-2u\rangle\, du$$

$$= \langle 0,0,6\rangle + \langle 3t, 4t, t-t^2\rangle$$

$$\boxed{\vec{r}(t) = \langle 3t, 4t, 6+t-t^2\rangle}$$

when does it hit the ground ?

set $6+t-t^2$ to zero; $t^2-t-6=0$, $(t-3)(t+2)=0$.

$t=3$, or $\cancel{t=-2}$ negative but t is a time.

$$\boxed{\vec{r}(3) = \langle 9, 12, 0\rangle}$$

- **Example 4:** Consider two moving particles whose positions at time $t \geq 0$ are given by

$$\vec{r_1}(t) = (t+1, 2t-1, 3t-4),$$
$$\vec{r_2}(t) = (3t, 2t^2+1, -t^2+3),$$

1. Determine whether the particles collide.

set $t+1 = 3t$, $\quad 2t-1 = 2t^2+1$, $\quad 3t-4 = -t^2+3$

$2t = 1$ $\qquad 0 \overset{?}{=} \frac{1}{2}+1$

$t = \frac{1}{2}$ $\qquad$ NO.

NO. They don't collide.

2. Find the intersection points of their paths.

set: $t+1 = 3s$, $2t-1 = 2s^2+1$, $3t-4 = -s^2+3$

$\boxed{\text{First equation}}$ $t+1 = 3s \Rightarrow 2t+2 = 6s$ $\Longrightarrow$ $3 = 6s - 2s^2 - 1$

$\boxed{\text{second equation:}}$ $2t-1 = 2s^2+1$

$2s^2 - 6s + 4 = 0$

$s^2 - 3s + 2 = 0$

$(s-1)(s-2) = 0$

For $s=1$, $t=2$. $\quad$ plugging these in the third equation:

$6-4 \overset{\checkmark}{=} -1+3$

For $s=2$, $t=5$ $\quad$ plugging these in the third equation:

$15-4 \neq -4+3$

Thus, the trajectories intersect for $t=2$, $s=1$, at the point:

$\vec{r_1}(2)\ (2+1, (2)(2)-1, (3)(2)-4)$

$= \boxed{(3, 3, 2)}$

- **Example 5:** Consider two moving particles whose positions at time $t \geq 0$ are given by

$$\vec{r_1}(t) = (3t, 3 + 4t, 1 + 2t),$$
$$\vec{r_2}(t) = (3 + 2t, 2 + 3t, 5 + 2t),$$

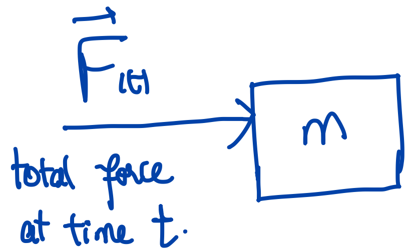
[a)]

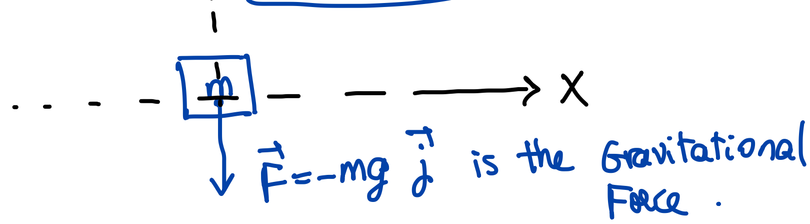
1. Find the minimum distance between the two particles.

2. Find the shortest distance between the trajectories of the two particles.

3. **Newton's Second Law of Motion:** If $F(t)$ is the force acting on a moving particle, and $a(t)$ the acceleration of the particle, then

$$\boxed{F(t) = ma(t)}.$$





4. **Example:** A projectile is fired with angle of elevation α and initial velocity v_0. Assuming the air resistance is negligible, and the only external force is due to gravity, find the position function $r(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

5. **Definition:** Suppose an object is moving in a circular path:

$$r(t) = a\cos(\theta(t))\vec{i} + a\sin(\theta(t))\vec{j}.$$

Then we define the angular speed by $w(t) := \frac{d\theta}{dt}$.