For a function of one variable: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

2 Section 14.3: Partial Derivatives

- Let $f(x, y)$ be a function of two variables. The partial derivatives of f with respect to x is:

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The partial derivative of f with respect to y is:

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

- Let $g(x, y, z)$ be a function of three variables. The partial derivative of g with respect to x is:

$$g_x(x, y, z) = \frac{\partial g}{\partial x}(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

The partial derivative of g with respect to y is:

$$g_y(x, y, z) = \frac{\partial g}{\partial y}(x, y, z) = \lim_{h\to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$$

The partial derivative of g with respect to z is:

$$g_z(x, y, z) = \frac{\partial g}{\partial z}(x, y, z) = \lim_{h\to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$$

Interpretation: $\frac{\partial f}{\partial x}$ is how much f increases as you move in the x-direction in one unit of the x-axis.

Similar statement can be made for the y and z directions.

Note: The definition implies that, to compute partial derivatives, we can treat all other variables as constant and differentiate as usual.

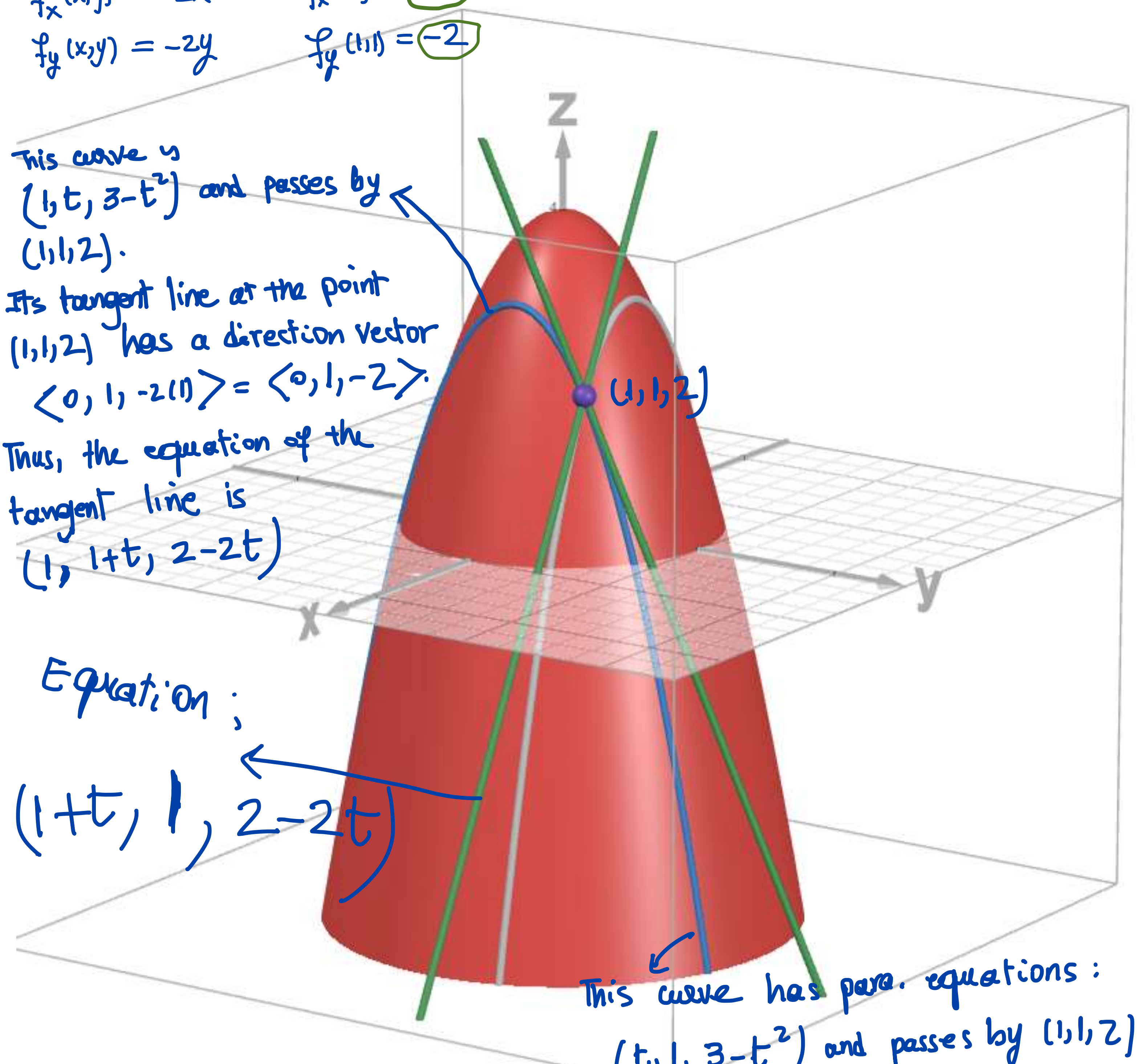
point $(1,1,2)$

- **Interpretations of Partial Derivatives**

  Example: Let $f(x,y) = 4 - x^2 - y^2$. Find $f_x(1,1)$ and $f_y(1,1)$ and interpret thse numbers as slopes.

$f_x(x,y) = -2x \qquad f_x(1,1) = -2$

$f_y(x,y) = -2y \qquad f_y(1,1) = -2$



This curve is $(1, t, 3-t^2)$ and passes by $(1,1,2)$.
Its tangent line at the point $(1,1,2)$ has a direction vector $\langle 0, 1, -2(1)\rangle = \langle 0, 1, -2\rangle$.
Thus, the equation of the tangent line is $(1, 1+t, 2-2t)$

Equation:

$(1+t, 1, 2-2t)$

This curve has para. equations: $(t, 1, 3-t^2)$ and passes by $(1,1,2)$
Its tangent line at the point $(1,1,2)$ has a direction vector $\langle 1, 0, (-2)(1)\rangle = \langle 1, 0, -2\rangle$.
Thus, the equation of the tangent line is $(1+t, 1, 2-2t)$

Solution:

- **Higher Order Derivatives**
  - If f is a function of two or three variables, then so are f_x, f_y, and f_z. So, we consider their partial derivatives as well.
  - **Example:** $f(x,y) = 7x^3y^2 - 5x^2y - y^3$

$f_x(x,y) = 21x^2y^2 - 10xy$; $f_y = 14x^3y - 5x^2 - 3y^2$; you can also compute

$f_{xy} = 42x^2y - 10x$ $\quad$ $f_{yx} = 42x^2y - 10x$ $\quad$ f_{xx}, f_{yy} - - -

Notice: $f_{xy} = f_{yx}$

  - **Notice:** In the above example $\boxed{f_{xy} = f_{yx}.}$ This is not a coincidence!
  - **Clairaut's Theorem:** Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

  - **Partial Differential Equations**
    - **Laplace's Equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions of this equation are called **harmonic functions** and appear in heat conduction, fluid flow, electric potential problems...
Ex: Show that $u(x,y) = e^x \sin(y)$ solves the Laplace's equation.

$u_x = e^x \sin(y)$ $\quad$ $u_{xx} = e^x \sin(y)$

$u_y = e^x \cos(y)$ $\quad$ $u_{yy} = -e^x \sin(y)$. $\quad$ So, $\boxed{u_{xx} + u_{yy} = 0}$

$u_t = -a\cos(x-at)$

$u_{tt} = (-a)(-a)\left(-\sin(x-at)\right) = -a^2 \sin(x-at)$

    - **Wave Equation** $\quad$ $u_x = \cos(x-at)$, $u_{xx} = -\sin(x-at)$.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

This equation discribes the motion of a waveform, which could be an ocean, sound, light wave...a wave traveling along a vibrating string...
Ex: Verify that $u(x,t) = \sin(x - at)$ satisfies the wave equation.