3 Section14.4: Tangent Planes and Linear Approximations

- Last time, we tried to see what the first partial derivatives represent graphically. Here is another example to recall what we found:

  Let $f(x,y) = x^2 + 2y^2 - 3$. Find $f_x(1,1)$ and $f_y(1,1)$, then explain what they represent graphically. $f(1,1) = 0$

Curve intersection of the surface with the plane $x=1$:

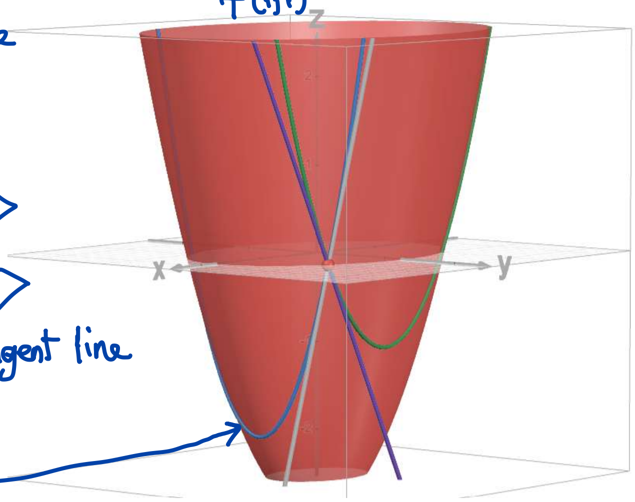
$r(t) = (1, t, 2t^2 - 2)$

Tangent vector $r'(t) = \langle 0, 1, 4t \rangle$

For $t=1$: $r'(1) = \langle 0, 1, 4 \rangle$

Parametric equations of the tangent line to the curve:

$L(t) = (1, 1+t, 4t)$



Curve intersection of the surface with the plane $y=1$:

$r(t) = (t, 1, t^2 - 1)$

Tangent vector $r'(t) = \langle 1, 0, 2t \rangle$

For $t=1$: $r'(1) = \langle 1, 0, 2 \rangle$

Parametric equations of the tangent line to the curve:

$L(t) = (1+t, 1, 2t)$

We see that:

- $f_y(1,1) = 4$ tells us that the tangent line to the parametric curve $(1, t, 2t^2 - 2)$ at the point $(1,1,0)$ has the parametric equation $(1, 1+t, 4t)$.
- $f_x(1,1) = 2$ tells us that the tangent line to the parametric curve $(t, 1, t^2 - 1)$ at the point $(1,1,0)$ has the parametric equation $(1+t, 1, 2t)$.
- It looks like any other tangent to a curve lives on the surface of f and passes by the point $(1,1,0)$ will be in the plane formed by the two tangents we just found! We can write the equation of such a plane! How ?

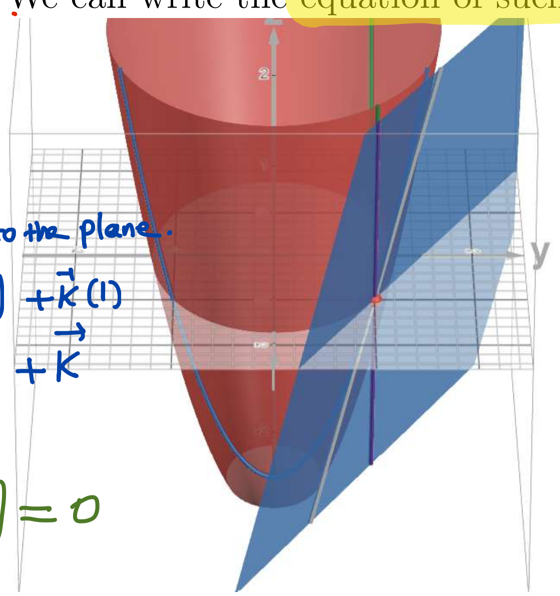
The point is $(x_0, y_0, f(x_0, y_0))$

The direction vectors of the two tangent lines we found earlier are:

$\vec{u} = (1, 0, f_x)$ and $\vec{v} = (0, 1, f_y)$.

Let $\vec{n} = \vec{u} \times \vec{v}$. Then $\vec{n}$ is a vector normal to the plane.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \vec{i}(-f_x) - \vec{j}(f_y) + \vec{k}(1) = -f_x\,\vec{i} - f_y\,\vec{j} + \vec{k}$$



Equation of the tangent plane:

$$-f_x(x - x_0) - f_y(y - y_0) + 1 \cdot (z - f(x_0, y_0)) = 0$$

Re-writing this as:

$$\boxed{z = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0).}$$

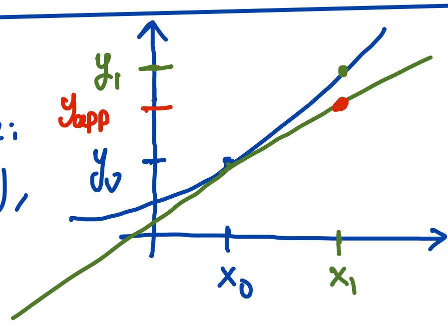
- Define a surface by a differentiable function $f(x, y)$, and let (x_0, y_0) be a point in the domain of f. The equation of the **tangent plane** to the surface at $f(x_0, y_0)$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(The proof is in the previous page).

Linear Approximation:

Recall: for a function of one variable: $y = f(x)$. Given a point (x_0, y_0), to approximate $f(x_1)$ we plug x_1 in the equation of the tangent line to the graph at the point (x_0, y_0).



For functions of TWO variables, we plug x_1 and y_1 in the equation of the tangent plane (see details below).

- The **Tangent plane** to the surface $z = f(x, y)$ at (x_0, y_0) is also called **linear approximation** for $f(x, y)$ around (x_0, y_0).

- **Example:** For $f(x, y) = e^{5-3x+7y}$, approximate $f(5.1, 1.9)$ using linear approximation at the point $(5, 2)$.

  Now use your calculator to check how big the error is.

Let $(x_0, y_0) = (5, 2)$. Let's find the equation of the tangent plane to the surface of f at the point $(5, 2)$.

$f(5,2) = e^{5-(3)(5)+(7)(2)} = e^4$

$f_x = -3e^{5-3x+7y}$, $f_y = 7e^{5-3x+7y}$

$f_x(5,2) = -3e^4$, $f_y(5,2) = 7e^4$

Equation of the tangent plane:

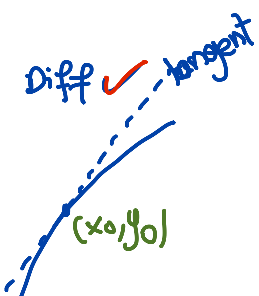
$z = f(5,2) + f_x(5,2)(x-5) + f_y(5,2)(y-2)$
$= e^4 - 3e^4(x-5) + 7e^4(y-2)$

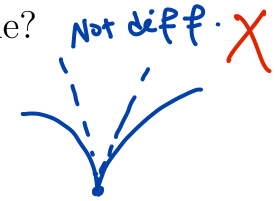
Linear app. of $f(5.1, 1.9)$ is

$e^4 - 3e^4(5.1-5) + 7e^4(1.9-2) \boxed{= 0}$

- **Differentiability**

  - What does it mean for a function of one variable $y = f(x)$ to be differentiable?





$\frac{\Delta y}{\Delta x}$ approaches a limit "a" as Δx goes to zero.

That is,

$\frac{\Delta y}{\Delta x} \approx a + \varepsilon \rightarrow$ error which goes to zero as $\Delta x \to 0$

Exact value: $\Delta y + y_0 \approx a\Delta x + y_0 + \varepsilon\Delta x$ Linear approximation

This says that the linear approximation gets closer and closer to the exact value as $\Delta x \to 0$

- What does it mean for a function of **two** variables $z = f(x, y)$ to be differentiable at (a, b)?

$$\underbrace{\Delta z + z_0}_{\text{Exact value}} = \underbrace{\left(f_x \Delta x + f_y \Delta y + z_0\right)}_{\text{Linear approximation.}} + \varepsilon\left(\Delta x + \Delta y\right)$$

Define

$$\Delta z := f(a + \Delta x, b + \Delta y) - f(a, b)$$

We say that f is differentiable at (a, b) if Δz can be expressed in the form:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_1 go to zero as Δx and Δy go to zero.

Good news: We have a Theorem which is easier to apply:

- **Theorem** If the partial derivatives f_x nd f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

- **Differentials**
  - For a differentiable function of one variable $y = f(x)$, we define the differential dx to be an independent variable, and the differential

$$dy = f'(x)dx.$$

  - For a differentiable function of two variable $z = f(x, y)$, we define the differential dx and dy to be the independent variables, and

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

- **Note:** For $dx = x - a$, and $dy = y - b$, then $dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}(x-a) + \frac{\partial z}{\partial y}(y-b)$, and

$$P(x,y) = f(a,b) + dz$$

is the linear approximation of f around (a,b).