- **Chain rule in the case of one independent variable t**

$f(x_{(t)})$

t

$f: \mathbb{R} \to \mathbb{R}$

  - In the case of single variable calculus, we have the chain rule:

$$\frac{d}{dt} f(g_{(t)}) = f'(g_{(t)}) \cdot g'_{(t)}. \quad \frac{d}{dt}f(g(t)) = f'(g(t))g'(t).$$

In other words: $df = f'_{(x)}\, dx$ Then, $\frac{df}{dt} = f'(x_{(t)}) \cdot \frac{dx}{dt}$

$f: \mathbb{R}^2 \to \mathbb{R}$

(x, y)

  - In the case of multi-variable calculus, we have the chain rule:

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

$dz = f_x\, dx + f_y\, dy$, then $\boxed{\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}}$

we write here:

$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

$\frac{df}{dt}$

  - **Example:** Let $z = f(x, y) = 5x^2 - 4y^2$, $x(t) = \sin(t)$, and $y(t) = \cos(t)$.
    - $*$ Compute $\frac{dz}{dt}$

$f_x = 10x$, $f_y = -8y$

$\frac{dx}{dt} = \cos(t)$, $\frac{dy}{dt} = -\sin(t)$.

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$$= (10x)(\cos(t)) - 8y(-\sin(t))$$

$$\frac{dz}{dt} = 10x\cos(t) + 8y\sin(t)$$

$$= (10)(\sin(t))(\cos(t)) + 8(\cos(t))(\sin(t)) = \boxed{18\sin(t)\cos(t)}$$

    - $*$ Evaluate $\frac{dz}{dt}$ at $t = 1$.

$$\frac{dz}{dt}(1) = 18\sin(1)\cos(1)$$

- **Chain rule in the case of two independent variable t**
  - In the case of single variable calculus, we have the chain rule:

$$\frac{\partial f(g(t,s))}{\partial t} = f'(g(t,s))\frac{\partial g(t,s)}{\partial t}$$

and

$$\frac{\partial f(g(t,s))}{\partial s} = f'(g(t,s))\frac{\partial g(t,s)}{\partial s}$$

  - In the case of multi-variable calculus, we have the chain rule:

$$\frac{d}{dt}f(x(t,s),y(t,s)) = f_x(x(t),y(t))\frac{\partial x(t,s)}{\partial t} + f_y(x(t),y(t))\frac{\partial y(t,s)}{\partial t}.$$

and

$$\frac{d}{ds}f(x(t,s),y(t,s)) = f_x(x(t),y(t))\frac{\partial x(t,s)}{\partial s} + f_y(x(t),y(t))\frac{\partial y(t,s)}{\partial s}.$$

  - **Example:** Let $z = f(x,y) = 5x^2 + xy - 3y^2$, $x(t) = 2u - 3v$, and $y(t) = u - 2v$. Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

Think of z now as a function of u and v instead of x and y.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (10x+y)(2) + (x-6y)(1)$$

$$= 20x + 2y + x - 6y$$

$$= 21x - 4y$$

$$= (21)(2u-3v) - 4(u-2v)$$

$$= 42u - 63v - 4u + 8v$$

$$\boxed{\frac{\partial z}{\partial u} = 38u - 55v}$$

$\frac{\partial z}{\partial x} = 10x + y$, $\frac{\partial z}{\partial y} = x - 6y$

$\frac{\partial x}{\partial u} = 2$, $\frac{\partial y}{\partial u} = 1$.

Find $\frac{\partial z}{\partial v}$ in a similar way.

- **Chain rule in the case of many independent variable t and for many-variables functions**

  Let $x = f(x_1(t_1, ..., t_n), x_2(t_1, ..., t_n), .., x_m(t_1, ..., t_n))$, where each of the $x_i(t_1, ..., t_n)$ is a differentiable function of $t_1, ..., t_n$.

  Then, z is a differentiable function of $t_1, ..., t_n$ and

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_j} + ... + \frac{\partial z}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

  For all $j = 1, .., n$.

- **Implicit Differentiation**

  $F : \mathbb{R}^2 \to \mathbb{R}$

  - Given an equation $\boxed{F(x, y) = 0}$, which defines y implicitly as a function of x, can we compute $\frac{dy}{dx}$ without explicitly solving for y in terms of x?

$$F(x,y) = 0 \; ; \quad \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = 0$$

Let $t = x$, then $\underbrace{\frac{\partial F}{\partial x}}_{F_x}\cdot 1 + \underbrace{\frac{\partial F}{\partial y}}_{=F_y}\frac{dy}{dx} = 0.$

then $F_y \frac{dy}{dx} = -F_x$. Then, $\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$

  - **The Implicit Function Theorem (for a function of two variables):** Let $F(x, y)$ be a function of two variable. If F is defined on a disk containing (a, b) where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$, defines y implicitly as a function of x near the point (a, b), and the derivative of the function is given by

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}.}$$

  - **Example:** Find $\frac{dy}{dx}$ if $x^4 + 5y^3 = 6x^2y$. around $(a,b) = (1,1)$

$$x^4 + 5y^3 - 6x^2y = 0$$

Let $F(x,y) = x^4 + 5y^3 - 6x^2y$

so, $F_x = 4x^3 - 12xy$ continuous

$F_y = 15y^2 - 6x^2$ continuous.

$(15)(1) - 6(1) \neq 0$

when $F_y \neq 0$, then

y is an implicit function of x and

$$\boxed{\frac{dy}{dx} = -\frac{4x^3 - 12xy}{15y^2 - 6x^2}}$$

- **The Implicit Function Theorem (for a function of three variables):** Let ~~$F(x,y)$~~ $F(x,y,z)$ be a function of three variable. If F is defined within a sphere containing (a,b,c) where $F(a,b,c)=0$, $F_z(a,b,c)\neq 0$, and F_x, F_y and F_z are continuous inside the sphere, then the equation $\boxed{F(x,y,z)=0,}$ defines z implicitly as a function of x and y near the point (a,b,c) , and the derivative of the function is given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

why?

$$F(x,y,z) = 0 \qquad F_x \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{=0} + F_z \cdot \frac{\partial z}{\partial x} = 0$$

?

so, $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

- **Example:** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$2x^3 - 3y^3 + 2z^3 + xyz - yz + 2 = 0$$

$$F(x,y,z) = 0$$

$F_x = 6x^2 + yz$; $F_y = -9y^2 + xz - z$;

$F_z = 6z^2 + xy - y$

so, $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{6x^2+yz}{6z^2+xy-y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-9y^2+xz-z}{6z^2+xy-y}.$$