MATH 215 WINTER 2026
Homework Set 4: Minimal Solutions

Question 1: Let $f(x,y) = xe^{-y^2} - ye^{-x^2}$.

(a) Find the equation for the tangent plane to the graph of f at the point $(2,1)$.

Solution: Generally speaking, for a function of the form $z = f(x,y)$, the tangent plane at the point (x_0, y_0, z_0) is given by

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

For this function, we have

$$f_x = e^{-y^2} + 2xye^{-x^2} \qquad\qquad f_y = -2xye^{-y^2} - e^{-x^2}$$

At the point $(2,1)$ we find

$$f(2,1) = 2e^{-1} - e^{-4} \qquad\qquad f_x(2,1) = e^{-1} + 4e^{-4} \qquad\qquad f_y(2,1) = -4e^{-1} - e^{-4}$$

Which gives our tangent plane approximation as

$$z - 2e^{-1} + e^{-4} = (e^{-1} + 4e^{-4})(x - 2) - (4e^{-1} + e^{-4})(y - 1)$$

(b) If one exists, find a point on the surface $z = x^2 - y^2$ that has a tangent plane parallel to the plane found in the previous part. If one does not exist, justify why.

Solution: We need to find a point on $z = f(x,y) = x^2 - y^2$ where the normal vector of the tangent plane is parallel to the normal vector of the tangent plane from part (a).
For the function $f(x,y) = x^2 - y^2$, the partial derivatives are simple to find as $f_x = 2x$ and $f_y = -2y$. The tangent plane, generically for a function of this type, will have normal vector

$$\mathbf{n} = \langle -f_x,\, -f_y,\, 1\rangle = \langle -2x,\, -2y,\, 1\rangle\,,$$

which, for this example, gives

$$\mathbf{n} = \left\langle e^{-1} + 4e^{-4},\, -4e^{-1} - e^{-4},\, 1\right\rangle$$

Then we see we must have

$$x = \frac{1}{2e} + \frac{2}{e^4}, \qquad y = \frac{2}{e} + \frac{1}{2e^4}, \qquad z = \left(\frac{1}{2e} + \frac{2}{e^4}\right)^2 - \left(\frac{2}{e} + \frac{1}{2e^4}\right)^2 = -\frac{15(e^6 - 1)}{4e^8}$$

Question 2: A function of two variables $u = u(x,t)$ is said to satisfy the *wave equation* in one space dimension if it satisfies the identity $u_{tt} = c^2 u_{xx}$. Here $c > 0$ is a constant denoting the speed of propagation of the wave.

(a) Take f and g to be two twice-differentiable functions of one variable. Show that

$$u(x,t) = f(x - ct) + g(x + ct)$$

is a solution of the wave equation.

Solution: The time derivatives come out to be

$$u_t = -cf'(x - ct) + cg'(x + ct), \qquad u_{tt} = c^2 f''(x - ct) + c^2 g''(x + ct)$$

and the spatial derivatives come out to be

$$u_x = f'(x - ct) + g'(x + ct), \qquad u_{xx} = f''(x - ct) + g''(x + ct)$$

We can see simply enough that $u_{tt} = c^2 u_{xx}$, as expected.

(b) One can show (but you don't have to) that all solutions of the one dimensional wave equation are of the above form for *some* f and g. Use this fact to find the solution of the wave equation that satisfies the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = xe^{-x^2/2}$$

Solution: The initial condition on the function implies

$$u(x,0) = 0 \qquad \Rightarrow \qquad f(x) + g(x) = 0 \qquad \Rightarrow \qquad g(x) = -f(x)$$

Our prospective solution can then be simplied to

$$u(x,t) = f(x - ct) - f(x + ct)$$

This has a temporal derivative of

$$u_t = -cf'(x - ct) - cf'(x + ct),$$

which, evaluated at $t = 0$, should be

$$u_t(x,0) = -cf'(x) - cf'(x) = -2cf'(x) = xe^{-x^2/2}$$

Isolating $f'(x)$ and integrating once we find

$$f'(x) = -\frac{x}{2c}e^{-x^2/2} \qquad \Rightarrow \qquad f(x) = \frac{1}{2c}e^{-x^2/2} + K$$

where K is an unknown constant of integration. (This constant is unimportant because it will cancel out when we put this all together for u.) Putting in our solution for f, we then have

$$u(x,t) = \frac{1}{2c}\left(e^{-(x-ct)^2/2} - e^{-(x+ct)^2/2}\right)$$

(c) Determine which, if any, of the following functions are solutions to Laplace's equation $u_{xx}+u_{yy}=0$:

$$f(x,y)=\frac{y}{a^2y^2-x^2} \qquad g(x,y)=e^{-x}\cos y-e^{-y}\cos x \qquad h(x,y)=\ln\sqrt{x^2+y^2}$$

Solution: There's no trick to this one, just a lot of derivatives. Let's get to it!
For f, the first and second derivatives are given by

$$f_x=\frac{2xy}{(a^2y^2-x^2)^2} \qquad f_y=\frac{-x^2-a^2y^2}{(a^2y^2-x^2)^2}$$

$$f_{xx}=\frac{2a^2y^3+6x^2y}{(a^2y^2-x^2)^3} \qquad f_{yy}=\frac{2a^4y^3+6a^2x^2y}{(a^2y^2-x^2)^3}$$

The sum is

$$f_{xx}+f_{yy}=\frac{2a^2y^3+6x^2y}{(a^2y^2-x^2)^3}+\frac{2a^4y^3+6a^2x^2y}{(a^2y^2-x^2)^3}=\frac{(1+a^2)(2a^2y^3+6x^2y)}{(a^2y^2-x^2)^3}\neq 0$$

which means this function *is not* a solution to Laplace's equation.
For g, the first and second derivatives are given by

$$g_x=-e^{-x}\cos y+e^{-y}\sin x \qquad g_y=-e^{-x}\sin y+e^{-y}\cos x$$

$$g_{xx}=e^{-x}\cos y+e^{-y}\cos x \qquad g_{yy}=-e^{-x}\cos y-e^{-y}\cos x$$

The sum is

$$g_{xx}+g_{yy}=e^{-x}\cos y+e^{-y}\cos x-e^{-x}\cos y-e^{-y}\cos x=0$$

and so this function *is* a solution.
Finally, for h it is first useful to do a bit of algebra and save ourselves some hassle with the derivatives. We notice

$$h(x,y)=\ln\sqrt{x^2+y^2}=\frac{1}{2}\ln\left(x^2+y^2\right)$$

The first and second derivatives are given by

$$h_x=\frac{x}{x^2+y^2} \qquad h_y=\frac{y}{x^2+y^2}$$

$$h_{xx}=\frac{y^2-x^2}{(x^2+y^2)^2} \qquad h_{yy}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

The sum is

$$h_{xx}+h_{yy}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0$$

and so this function *is* a solution.

Question 3:

(a) Newton's law of universal gravitation states that the magnitude of the gravitational force F between two objects is given by

$$F = G\frac{m_1 m_2}{r^2},$$

where G is the gravitational constant, m_1 and m_2 are the masses of the two objects, and r is the distance between the objects. Here $G \approx 6.674 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$. A team of amateur astronomers have estimated that $m_1 = 2 \times 10^{24}$ kg, $m_2 = 5 \times 10^{23}$ kg, and $r = 10^{10}$ m, with a maximum relative errorof 3% in each measurement. Use differentials to estimate the maximum relative error in the calculated force F.

Solution: Using the chain rule (or computing the total differential, which is more or less the same process), we find

$$dF = \frac{\partial F}{\partial m_1}\,dm_1 + \frac{\partial F}{\partial m_2}\,dm_2 + \frac{\partial F}{\partial r}\,dr$$

In order to compute the relative error, we're going to want dF/F. Before we write that down, though, let's first collect our partial derivatives:

$$\frac{\partial F}{\partial m_1} = \frac{Gm_2}{r^2} \qquad \frac{\partial F}{\partial m_2} = \frac{Gm_1}{r^2} \qquad \frac{\partial F}{\partial r} = -\frac{2Gm_1 m_2}{r^3}$$

Notice, then, that the terms in our relative error will have particularly nice forms:

$$\frac{1}{F}\frac{\partial F}{\partial m_1} = \frac{1}{m_1} \qquad \frac{1}{F}\frac{\partial F}{\partial m_2} = \frac{1}{m_2} \qquad \frac{1}{F}\frac{\partial F}{\partial r} = -\frac{2}{r}$$

The relative error is then given as

$$\frac{dF}{F} = \frac{dm_1}{m_1} + \frac{dm_2}{m_2} - 2\frac{dr}{r}$$

Notice each term in this sum is exactly the relative error in each of the variables. Because we don't know if the errors in measurement are over- or under-estimates, we need to allow for the possibility that all of the errors could work in concert. That is,

$$\left|\frac{dF}{F}\right| \leq \left|\frac{dm_1}{m_1}\right| + \left|\frac{dm_2}{m_2}\right| + \left|-2\frac{dr}{r}\right| = 3\% + 3\% + 2\,(3\%) = 12\%$$

(b) Use differentials to approximate the number $(1.98)^3\left((3.03)^2 - \frac{1}{(1.01)^3}\right)$. It may help to consider a suitable function $f(x, y, z)$ at a suitable point $P(a, b, c)$.

Solution: Let our function be given by

$$f(x, y, z) = x^3\left(y^2 - \frac{1}{z^3}\right)$$

at the point $(2, 3, 1)$. The partial derivatives are given by

$$f_x = 3x^2\left(y^2 - \frac{1}{z^3}\right), \qquad f_y = 2x^3y, \qquad f_z = \frac{3x^3}{z^4}$$

At the point in question, our function and its partial derivatives have the values

$$f(2, 3, 1) = 64, \;\; f_x(2, 3, 1) = 96, \;\; f_y(2, 3, 1) = 48, \;\; f_z(2, 3, 1) = 24$$

The linearization is given by

$$f(x, y, z) \approx f(2, 3, 1) + f_x(2, 3, 1)\,(x - 2) + f_y(2, 3, 1)(y - 3) + f_z(2, 3, 1)(z - 1)$$

And so

$$\begin{aligned} f(1.98,\ 3.03,\ 1.01) &\approx 64 + 96\,(-0.02) + 48\,(0.03)) + 24\,(0.01) \\ &= 64 - \frac{192}{100} + \frac{144}{100} + \frac{24}{100} \\ &= 64 - \frac{24}{100} \\ &= 64 - 0.24 \\ &= 63.76 \approx 63.73164351324\ldots \end{aligned}$$

so this approximation is good to the first decimal place, and falls off a bit after that.

Question 4: Suppose $f(x, y)$ is a twice continuously differentiable function with function values measured in the table below:

y \ x	-1	0	1	2	3
-1	11	12	15	14	13
0	13	16	17	18	20
1	20	22	22	19	18
2	27	26	25	22	20
3	32	28	28	27	26

(a) Approximate f_x and f_y at the point $(1, 2)$.

Solution: We're going to approximate these using a *center difference* formula. Suppose we have a partition of our domain where Δx and Δy are uniform across the partition. (If the step sizes are not uniform then some extra care must be taken.) Then we would define the *forwards* *difference* approximation as

$$f_x \approx \frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{\Delta x}$$

and analogously for f_y. The *backwards difference* approximation would be

$$f_x \approx \frac{f(x_i, y_j) - f(x_{i-1}, y_j)}{\Delta x}$$

and the *center difference* approximation is

$$f_x \approx \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{2\Delta x}$$

Generally speaking, the center difference approximation will have better convergence properties, and so we would like to use it whenever possible. For this particular example, notice that $\Delta x = \Delta y = 1$ everywhere in the partition. Then our approximation of the x derivative requires us to look at the two orange entries below (we have colored the point about which we wish to approximate in blue):

y \ x	-1	0	1	2	3
-1	11	12	15	14	13
0	13	16	17	18	20
1	20	22	22	19	18
2	27	26	25	22	20
3	32	28	28	27	26

This gives

$$f_x(1,2) \approx \frac{f(2,2) - f(0,2)}{2} = \frac{22 - 26}{2} = -2$$

To approximate the partial derivative with respect to y, we apply the same process, except moving in the y-direction. In the table below, this means we look at the values of the function in the two orange cells marked below:

y \ x	-1	0	1	2	3
-1	11	12	15	14	13
0	13	16	17	18	20
1	20	22	22	19	18
2	27	26	25	22	20
3	32	28	28	27	26

This gives

$$f_y(1,2) \approx \frac{f(1,3) - f(1,1)}{2} = \frac{28 - 22}{2} = 3$$

(b) Approximate f_{xy} and f_{xx} at the point $(1, 2)$.

Solution: We're going to apply the same methodology here as we did in the last part, which is going to require repeating part (a) a few different times. We'll have

$$f_{xx}(1,2) \approx \frac{f_x(2,2) - f_x(0,2)}{2}$$

In order to find $f_x(2,2)$ and $f_x(0,2)$ we're going to need to approximate them the same as we did in the last part. We have

$$f_x(2,2) \approx \frac{f(3,2) - f(1,2)}{2} = -\frac{5}{2} \qquad f_x(0,2) \approx \frac{f(1,2) - f(-1,2)}{2} = -1$$

which will give

$$f_{xx}(1,2) \approx \frac{f_x(2,2) - f_x(0,2)}{2} = \frac{1}{2}\left(-\frac{5}{2} - (-1)\right) = -\frac{3}{4}$$

Similarly, we have

$$f_{xy}(1,2) \approx \frac{f_x(1,3) - f_x(1,1)}{2}$$

which requires us to compute

$$f_x(1,3) \approx \frac{f(2,3) - f(0,3)}{2} = -\frac{1}{2} \qquad f_x(1,1) \approx \frac{f(2,1) - f(0,1)}{2} = -\frac{3}{2}$$

giving

$$f_{xy}(1,2) \approx \frac{f_x(1,3) - f_x(1,1)}{2} = \frac{1}{2}\left(-\frac{1}{2} - \left(-\frac{3}{2}\right)\right) = \frac{1}{2}$$

(c) Using the table directly, approximate the directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = \langle 1, -1 \rangle$.

Solution: We're going to apply the same method as twice before, except this time we're going to move in the direction $\mathbf{u}$, which is along the *diagonal*:

y \ x	-1	0	1	2	3
-1	11	12	15	14	13
0	13	16	17	18	20
1	20	22	22	19	18
2	27	26	25	22	20
3	32	28	28	27	26

Notice that we are *increasing* x by 1, but *decreasing* y by 1. Then we have

$$D_{\hat{u}} f(1,2) \approx \frac{f(2,1) - f(0,3)}{2\sqrt{2}} = -\frac{9}{2\sqrt{2}}$$

Notice that because we have moved along a diagonal, our distance between the partition points is not 1, but rather $\sqrt{2}$.

(d) Using the gradient vector, approximate the directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = \langle 1, -1 \rangle$. Does your answer agree with the previous part? Explain.

Solution: Because our function is sufficiently smooth, in principle we will have

$$D_{\hat{u}} f(1, 2) = \nabla f(1, 2) \cdot \hat{u}$$

Using our results from part (a), we have

$$f_x(1, 2) \approx -2 \qquad f_y(1, 2) \approx 3 \qquad \hat{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

Putting all of this together, we find our alternate approximation as

$$D_{\hat{u}} f(1, 2) \approx (-2) \cdot \left(\frac{1}{\sqrt{2}} \right) + (3) \cdot \left(-\frac{1}{\sqrt{2}} \right) = -\frac{5}{\sqrt{2}}$$

This is pretty close to the approximation we found in the last step! This is to be expected, as explained above.

Question 5: Consider the ellipsoid $x^2 + 2y^2 + 4z^2 + xy + 4yz = 71$.

(a) Show that the points on the ellipsoid where the tangent plane is vertical (parallel to the z-axis) constitute the intersection of the ellipsoid with a certain plane, and find the equation of that plane.

Solution: If we use the methods from section 14.6, we can compute the normal vector to the ellipsoid. The ellipsoid is the level set of a function $F(x, y, z) = x^2 + 2y^2 + 4z^2 + xy + 4yz$. Then normal vector to the tangent plane may be taken to be the gradient of F, giving

$$\mathbf{n} = \nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x + y, 4y + x + 4z, 8z + 4y \rangle$$

Vertical planes will have normal vectors with no z-component, i.e. $\mathbf{n} \parallel \langle \cdot, \cdot, 0 \rangle$. Thus, we know we need

$$8z + 4y = 0$$

and that is the plane that we are looking for.

If we haven't yet read section 14.6, we can still work out the answer, with slightly less precise, but still relatively convincing, work. In a vertical tangent plane, there will be a vector that points straight up–this vector will have infinite slope. (Importantly, vectors in non-vertical tangent planes cannot have this property.) By using implicit differentiation (exactly as we will do in part (b)), we can show that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

In order for these to be infinite (i.e. an infinite slope line in a vertical plane), we need the denominators to be zero, which brings us back to the condition we articulated before.

(b) Consider the point $P(1, 2, 3)$ (check that it is on the ellipsoid!). Since this point is not among those of part (a), a piece of the ellipsoid containing P is the graph of a function $g(x, y)$. Use implicit differentiation to compute g_x and g_y in terms of $(x, y, g(x, y))$, as well as $g_x(1, 2)$ and $g_y(1, 2)$.

Solution: First, the partial derivatives of F are given by

$$F_x = 2x + y, \qquad F_y = 4y + x, \qquad F_z = 8z + 4y$$

Using implicit differentiation, we know we will have

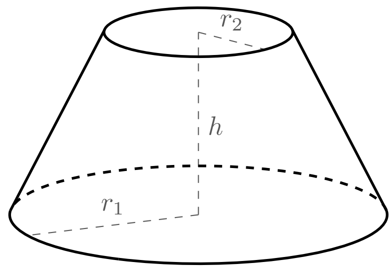
$$g_x = -\frac{F_x}{F_z} = -\frac{2x + y}{8z + 4y}, \qquad g_y = -\frac{F_y}{F_z} = -\frac{4y + x + 4z}{8z + 4y}$$

At the point of interest, we know $x = 1$, $y = 2$, and $z = 3$ giving

$$g_x(1, 2) = -\frac{4}{32} = -\frac{1}{8} \qquad g_y(1, 2) = -\frac{21}{32}$$

Question 6:

(a) A truncated right circular cone has a height, and two radii (see picture below). The smaller radius of this cone is decreasing at a constant rate of 1 cm/s, the larger radius is increasing at a constant rate of 2 cm/s, and the height of the cone is decreasing at a constant rate of 3 cm/s. At what rate is the volume of the cone changing when the smaller radius is 10 cm, the larger radius is 15 cm, and the height is 8 cm?



Solution: The volume of the truncated cone (more properly called a *frustum*) is just the volume of the larger cone minus the volume of the top part of the cone that has been chopped off. Let the height of the smaller cone be x. Congruent triangles will then give

$$\frac{r_1}{x+h} = \frac{r_2}{x} \qquad \Rightarrow \qquad x = \frac{r_2 h}{r_1 - r_2}, \quad x + h = \frac{r_1 h}{r_1 - r_2}$$

The volume is then given by

$$V = \frac{\pi}{3} r_1^2 (x+h) - \frac{\pi}{3} r_2^2 h = \frac{\pi h}{3}\left(\frac{r_1^3 - r_2^3}{r_1 - r_2}\right) = \frac{\pi h}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right)$$

Using the chain rule, we then have

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial r_1}\cdot\frac{dr_1}{dt} + \frac{\partial V}{\partial r_2}\cdot\frac{dr_2}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}\\ &= \left(\frac{(2r_1 + r_2)\pi h}{3}\right)\frac{dr_1}{dt} + \left(\frac{(r_1 + 2r_2)\pi h}{3}\right)\frac{dr_2}{dt} + \left(\frac{(r_1^2 + r_1 r_2 + r_2^2)\pi}{3}\right)\frac{dh}{dt}\end{aligned}$$

Everything in the above expression is given in the problem statement, so now it is just a matter of plugging the numbers in. We take $r_1 = 15$, $r_2 = 10$, $h = 8$, and $dr_1/dt = 2$, $dr_2/dt = -1$, $dh/dt = -3$ to find

$$\frac{dV}{dt} = -355\pi\,\frac{\text{cm}^3}{\text{s}}$$

(b) If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

Solution: There is no real trick to this problem, just a lot of details–the hardest part is keeping track of all of the derivatives. Our goal here is to repeatedly apply the chain rule to calculate the polar derivatives of z in terms of the cartesian derivatives of z.

The first two derivatives work out nicely:

$$z_r = z_x x_r + z_y y_r \qquad\qquad z_\theta = z_x x_\theta + z_y y_\theta$$

Naively applying the chain rule again, we find, for example

$$z_{rr} = z_{xr}x_r + z_x x_{rr} + z_{yr}y_r + z_y y_{rr}$$
$$z_{\theta\theta} = z_{x\theta}x_\theta + z_x x_{\theta\theta} + z_{y\theta}y_\theta + z_y y_{\theta\theta}$$

We can see immediately that we are going to need to find the mixed derivatives z_{xr}, z_{yr}, $z_{x\theta}$, and $z_{y\theta}$. However, this is also straightforward. Applying the chain rule the exact same way that we did before, we find

$$z_{xr} = z_{xx}x_r + z_{xy}y_r \qquad z_{x\theta} = z_{xx}x_\theta + z_{xy}y_\theta$$
$$z_{yr} = z_{yx}x_r + z_{yy}y_r \qquad z_{y\theta} = z_{yx}x_\theta + z_{yy}y_\theta$$

At this point, we can continue working with general derivatives, or we can make use of the fact that we know the derivatives of x and y. Both methods will work, but by using information about x and y now, we can eliminate some of the terms in our expressions above. Let's collect some information:

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$x_r = \cos\theta \qquad y_r = \sin\theta$$
$$x_\theta = -r\sin\theta \qquad y_\theta = r\cos\theta$$
$$x_{rr} = 0 \qquad y_{rr} = 0$$
$$x_{\theta\theta} = -r\cos\theta \qquad y_{\theta\theta} = -r\sin\theta$$

(We did not collect the mixed derivatives of x and y because they don't appear in any of our expressions, but make sure you know how to find them.) Plugging in this information to our expressions for the derivatives we need, we find the following:

$$z_r = z_x\cos\theta + z_y\sin\theta \qquad z_\theta = -z_x r\sin\theta + z_y r\cos\theta$$
$$z_{xr} = z_{xx}\cos\theta + z_{xy}\sin\theta \qquad z_{x\theta} = -z_{xx}r\sin\theta + z_{xy}r\cos\theta$$
$$z_{yr} = z_{yx}\cos\theta + z_{yy}\sin\theta \qquad z_{y\theta} = -z_{yx}r\sin\theta + z_{yy}r\cos\theta$$

Putting these into our expressions for the second polar derivatives, after simplification, we have

$$z_{rr} = z_{xx}\cos^2\theta + z_{xy}\cos\theta\sin\theta + z_{yx}\cos\theta\sin\theta + z_{yy}\sin^2\theta$$

and

$$\frac{1}{r^2}z_{\theta\theta} = z_{xx}\sin^2\theta - z_{xy}\cos\theta\sin\theta - \frac{z_x}{r}\cos\theta - z_{yx}\cos\theta\sin\theta + z_{yy}\cos^2\theta - \frac{z_y}{r}\sin\theta$$

Combining everything together we now find

$$\begin{aligned}
z_{rr} &= z_{xx}\cos^2\theta + z_{xy}\cos\theta\sin\theta + z_{yx}\cos\theta\sin\theta + z_{yy}\sin^2\theta \\
+\frac{1}{r^2}z_{\theta\theta} &= z_{xx}\sin^2\theta - z_{xy}\cos\theta\sin\theta - z_{yx}\cos\theta\sin\theta + z_{yy}\cos^2\theta - \frac{z_x}{r}\cos\theta - \frac{z_y}{r}\sin\theta \\
+\frac{1}{r}z_r &= \qquad\qquad + \frac{z_x}{r}\cos\theta + \frac{z_y}{r}\sin\theta
\end{aligned}$$

$$\begin{aligned}
&= z_{xx}\left(\cos^2\theta + \sin^2\theta\right) + z_{yy}\left(\sin^2\theta + \cos^2\theta\right) \\
&= z_{xx} + z_{yy}
\end{aligned}$$

and we have now established the desired equality.

Extra Credit: In this question let's explore how much more interesting the notion of continuity and differentiability can be in higher dimensions. First, let's look at continuity:

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ (this notation means all points in the plane except for the origin):

$$f(x,y) = \frac{xy}{x^2+y^2}$$

By letting $x = r\cos\theta$ and $y = r\sin\theta$, describe the level sets of f. Explain why there is no value we can assign to $f(0,0)$ that would make this function continuous.

Solution: Converting into polar coordinates we find

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$$

For constant values of θ (i.e. straight lines), this function will approach different values. For example, when $\theta = 0$, we have the line $y = 0$, and along this path we see

$$\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} 0 = 0$$

However, when $\theta = \pi/4$, we have the line $y = x$, and along this path we see

$$\lim_{(x,y)\to(0,0)} f(x,x) = \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

As $0 \neq 1/2$, it is not possible for us to define a value for f at the origin that will make it continuous. No matter how close we get to the origin, there is always a point nearby that has a value of 0 and always another point nearby that has a value of $1/2$.

(b) Now consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined everywhere on $\mathbb{R}^2$ by

$$g(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Using the same basic trick as in part (a), explain how you know

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

Solution: Applying the same basic methodology as before, we have

$$g(x,y) = \frac{xy}{\sqrt{x^2+y^2}} = \frac{r^2\cos\theta\sin\theta}{r} = r\cos\theta\sin\theta$$

Notice, then, that

$$|g(x,y)| = |r\cos\theta\sin\theta| \leq |r| = r$$

Next, we note that the origin is the only point in $\mathbb{R}^2$ for which $r = 0$, and so any limit that sends $(x,y) \to (0,0)$ is covered by a limit that sends $r \to 0$. Then we can use the squeeze theorem to argue the following:

$$\lim_{(x,y)\to(0,0)} |g(x,y)| = \lim_{r\to 0} |g(x,y)| = \lim_{r\to 0} |r\cos\theta\sin\theta| \leq \lim_{r\to 0} |r| = \lim_{r\to 0} r = 0$$

We have now shown that as $r \to 0$, $|g(x,y)| \to 0$, which means $g(x,y) \to 0$. Thus, the limit is equal to $g(0,0)$, and this function is continuous.

(c) One last interesting example. Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$h(x,y) = \frac{x^2 y}{x^4 + y^2}$$

Is there a value you can assign to $h(0,0)$ to make h continuous at the origin? Justify your work.

Solution: There is not!

The cool thing about this question is that every straight line path towards the origin approaches the same value. Let $y = ax$, for some value of a. Then

$$\lim_{(x,y)\to(0,0)} h(x,ax) = \lim_{x\to 0} \frac{x^2 (ax)}{x^4 + a^2x^2} = \lim_{x\to 0} \frac{ax^3}{x^4 + a^2x^2} = \lim_{x\to 0} \frac{ax}{x^2 + a^2} = 0$$

(I will leave it to you to verify that the line $x = 0$ also approaches this same value.)

However, if we approach the origin *not* along a straight line, we can get something different. Suppose $y = ax^2$, for some value of a. Then

$$\lim_{(x,y)\to(0,0)} h(x,ax^2) = \lim_{x\to 0} \frac{x^2 \left(ax^2\right)}{x^4 + a^2x^4} = \lim_{x\to 0} \frac{ax^4}{x^4 + a^2x^4} = \lim_{x\to 0} \frac{a}{1 + a^2} = \frac{a}{1 + a^2}$$

which is not equal to zero as long as $a \neq 0$.

Thus, this function is discontinuous at the origin, as the limit is not well-defined.

Now let's turn to differentiability.

(d) Now consider the function $p(x,y) = (xy)^{1/3}$. Compute $p_x(x,0)$ for any x and $p_y(0,y)$ for any y. In particular, compute both p_x and p_y at $(0,0)$.

Solution: Away from the lines $x = 0$ and $y = 0$, we can compute the partial derivatives in the standard, shortcut way:

$$p_x = \frac{\partial}{\partial x}\left[(xy)^{1/3}\right] = \frac{y^{1/3}}{3x^{2/3}} \qquad\qquad p_y = \frac{\partial}{\partial y}\left[(xy)^{1/3}\right] = \frac{x^{1/3}}{3y^{2/3}}$$

We can see immediately that these formula will fail if $x = 0$ or if $y = 0$, and so we are going to need to resort to some other method. (Technically speaking they will work as long as at least one of x or y is not zero, and because of the derivatives I asked you to find they will work here, but this is still a strong hint that we are going to need to try something else.) Specifically, we are going to use the definition of the partial derivative. We have

$$p_x(x,0) = \lim_{h\to 0} \frac{f(x+h,0) - f(x,0)}{h} = \lim_{h\to 0} \frac{((x+h)\cdot 0)^{1/3} - (x\cdot 0)^{1/3}}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

An entirely analogous calculation will show $p_y(0,y) = 0$.

In particular, the same calculation will show that $p_x(0,0) = p_y(0,0) = 0$.

(e) Along the positive x-axis, does this function have a tangent plane? What is it? What about along the positive y-axis?

Solution: In the previous part, notice that we computed $p_x(x,0)$ for any x, but we did *not* compute $p_y(x,0)$. If we go back to the definition and try this, we will find

$$p_y(x,0) = \lim_{h\to 0} \frac{f(x,h) - f(x,0)}{h} = \lim_{h\to 0} \frac{(x\cdot h)^{1/3} - (x)^{1/3}}{h^{2/3}} = \text{does not exist}$$

This suggests to us that we're going to have a vertical tangent plane of some kind, which you can easily visually verify.

If we wanted to show this more concretely, we recognize that $z = (xy)^{1/3}$ is the same equation as $z^3 = xy$. If we define $F(x, y, z) = z^3 - xy$, then our surface is the level set of $F = 0$, and we know that ∇F will be the normal vector to the tangent plane. We have

$$\nabla F = \langle y,\, x,\, 3z^2 \rangle$$

Along the positive x-axis, this gives $\nabla F = \langle 0,\, x,\, 0 \rangle$, which is a vector parallel to the y-axis, indicating a vertical tangent plane as we guessed. This tangent plane has equation $y = 0$.

Along the positive y-axis, we'll have $\nabla F = \langle y,\, 0,\, 0 \rangle$, again giving a vertical tangent plane of $x = 0$.

Because of the verticality, our method for computing tangent planes from section 14.4 fails: along these axes, z fails to be a function of x and y. The surface is perfectly nice (with one exception, discussed below), but our formula for the tangent plane doesn't apply because $z = f(x, y)$ fails the vertical line test, even if only at a point.

(f) Does this function have a tangent plane at the origin? Explain.

Solution: It does not. If we try to apply the same level set/gradient method as in the last part, we find $\nabla F = \langle 0,\, 0,\, 0 \rangle$, indicating a point of non-regularity. To show that this point is *actually* irregular (and this is not just an artifact of our choice of parametrization, to be discussed again in Chapter 16) is beyond the scope of this course. We can, however, gain some intuition based on our earlier calculations.

Visually, you can see that there is a corner (or a "pinch") in the surface at the origin. This means that we are not going to be able to nicely define a tangent plane at that point. How does this fact manifest itself in our earlier calculations? Suppose we are on the positive x-axis, near, but not at, the origin. Then we have a good definition of a tangent plane, given by $y = 0$. Similarly, if we are on the positive y-axis, near but not at the origin, we have a good definition of the tangent plane, given by $x = 0$. As we transition between these two points, we have no way to smoothly transition the plane $x = 0$ into the plane $y = 0$, because we can be arbitrarily close to the origin in either case.